

# First-principle Derivation of Stable First-order Generic-frame Relativistic Dissipative Hydrodynamic Equations from Kinetic Theory by Renormalization-group Method

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We derive first-order relativistic dissipative hydrodynamic equations from relativistic Boltzmann equation on the basis of the renormalization-group (RG) method. We introduce a macroscopic-frame vector (MFV), which does not necessarily coincide with the flow velocity, to specify the local rest frame on which the macroscopic dynamics is described. The five hydrodynamic modes are naturally identified with the same number of the zero modes of the linearized collision operator, i.e., the collision invariants. After defining the inner product in the function space spanned by the distribution function, the higher-order terms, which give rise to the dissipative effects, are constructed so that they are precisely orthogonal to the zero modes in terms of the inner product: Here, any ansatz's, such as the so-called conditions of fit used in the standard methods in an ad-hoc way, are not necessary. We elucidate that the Burnett term does not affect the hydrodynamic equations owing to the very nature of the hydrodynamic modes as the zero modes. Then, applying the RG equation, we obtain the hydrodynamic equation in a generic frame specified by the MFV, as the coarse-grained and covariant equation. Our generic hydrodynamic equation reduces to hydrodynamic equations in various local rest frames, including the energy and particle frames with a choice of the MFV. We find that our equation in the energy frame coincides with that of Landau and Lifshitz, while the derived equation in the particle frame is slightly different from that of Eckart, owing to the presence of the dissipative internal energy. We prove that the Eckart equation can not be compatible with the underlying relativistic Boltzmann equation. The proof is made on the basis of the observation that the orthogonality condition to the zero modes coincides with the ansatz's posed on the dissipative parts of the energy-momentum tensor and the particle current in the phenomenological equations. We also present an analytic proof that all of our equations have a stable equilibrium state owing to the positive definiteness of the inner product.

## §1. Introduction

Relativistic hydrodynamic equation is widely used in various fields of physics, especially in high-energy nuclear physics<sup>1)–3)</sup> and astrophysics,<sup>4)–6)</sup> and it seems that the study of the relativistic hydrodynamic equation with *dissipative* effects is now becoming a central interest in these fields.

For instance, it was shown that the dynamical evolution of the hot and/or dense QCD matter produced in the Relativistic Heavy Ion Collider (RHIC) experiments can be well described by the relativistic hydrodynamic simulations.<sup>2), 3)</sup> The suggestion that the created matter may have only a tiny viscosity prompted an interest in the origin of the viscosity in the created matter to be described by the relativistic quantum field theory and also the dissipative hydrodynamic equations. Also the relativistic dissipative hydrodynamic equation has been applied to the various high-energy astrophysical phenomena, e.g., the accelerated expansion of the universe by bulk viscosity of dark matter and/or dark energy.<sup>5), 6)</sup>

It is, however, noteworthy that we have not necessarily reached a full understanding of the theory of relativistic hydrodynamics for viscous fluids, although there have been many important studies since Eckart's pioneering work.<sup>11)</sup>

### 1.1. *Fundamental problems with relativistic hydrodynamic equation for a viscous fluid*

We may summarize the fundamental problems on the relativistic dissipative hydrodynamic equations as follows: (a) ambiguities and ad-hoc ansatz's in the definition of the flow velocity;<sup>8), 10)–16)</sup> (b) unphysical instabilities of the equilibrium state;<sup>17)</sup> (c) lack of causality.<sup>16), 18)–20)</sup>

One might consider that the definition of the flow velocity should be merely a kind of the choice of the coordinate space to describe dynamics in a easy and practical way, and one can go back and forth between two different definitions by a Lorentz transformation. Furthermore, one might consider that the unphysical instabilities of the equilibrium state should be attributable to the lack of causality, and one can restore the instabilities automatically by solving the causality problem. Thus, although the relativistic dissipative hydrodynamic equation has been attracting a great interest, it seems that most works are concerned with the causality problem and examine phenomenological or semi-phenomenological causal equations with some foundations. We argue, however, that the first two problems, (a) and (b), and the third one, (c), have different origins, and the first two must be resolved before the third one is addressed. The present paper is concerned with the first two problems, and the third one will be studied in the forthcoming paper.<sup>21)</sup>

(a) The form of the relativistic dissipative hydrodynamic equation depends on the definition of the flow velocity, which is equivalent to the choice of the local rest frame of the fluid. Typical rest frames include the particle frame and the energy frame, and a phenomenological equation for the respective frame is constructed by Eckart<sup>11)</sup> and Landau and Lifshitz,<sup>12)</sup> respectively. Here, the phenomenological construction is based on the following three ingredients; (A) the particle-number and energy-momentum conservation laws, (B) the law of the increase in entropy, and (C) some specific assumptions on the choice of the flow. The first and second points are reasonable and used also in the construction of the non-relativistic Navier-Stokes equation, while the third point is specific for the relativistic case. To be more explicit, let  $\delta T^{\mu\nu}$  and  $\delta N^\mu$  be the dissipative part of the energy-momentum tensor and the particle current, respectively. The point is that the forms of  $\delta T^{\mu\nu}$  and  $\delta N^\mu$  are not determined uniquely only by the particle-number and energy-momentum conservation laws and the law of the increase in entropy without some physical ansatz's involving the flow velocity  $u_\mu$  with  $u_\mu u^\mu = g^{\mu\nu} u_\mu u_\nu = 1$  and  $g^{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ : The implicit assumptions made by Eckart<sup>11)</sup> are

$$(i) \quad \delta e \equiv u_\mu \delta T^{\mu\nu} u_\nu = 0, \quad (1.1)$$

$$(ii) \quad \delta n \equiv u_\mu \delta N^\mu = 0, \quad (1.2)$$

$$(iii) \quad \nu_\mu \equiv \Delta_{\mu\nu} \delta N^\nu = 0, \quad (1.3)$$

where  $\Delta_{\mu\nu} \equiv g_{\mu\nu} - u_\mu u_\nu$ . The condition (i) claims the absence of the internal energy  $\delta e$  of the dissipative origin, while (ii) and (iii) require no dissipative particle-number

density  $\delta n$  and current  $\nu^\mu$ , respectively. On the other hand, those of Landau and Lifshitz<sup>12)</sup> consist of (i), (ii), and

$$(iv) \quad Q_\mu \equiv \Delta_{\mu\nu} \delta T^{\nu\rho} u_\rho = 0. \quad (1.4)$$

In physical terms, (i) and (iv) claim the absence of the internal energy  $\delta e$  and current  $Q^\mu$  of dissipative origin. As one sees, the first two ansatz's are common for the two equations, and the third ones ((iii) and (iv)) are supposed to specify the respective local rest frame of the flow velocity. We note that the conditions (iii) and (iv) and hence the two of frames specified by these conditions can not be connected with each other by a Lorentz transformation. It is here noteworthy that there is a proposal by Stewart<sup>14)</sup> for the condition for the particle frame, as given by (ii), (iii), and

$$(v) \quad \delta T^\mu{}_\mu = \delta e - 3 \delta p = 0, \quad (1.5)$$

where  $\delta p \equiv -\Delta_{\mu\nu} \delta T^{\mu\nu}/3$  is the dissipative pressure to be identified with the standard bulk pressure. Here, the condition (i) of Eckart is replaced by the different one (v), which claims a constraint between the dissipative internal energy  $\delta e$  and the dissipative pressure  $\delta p$ . One may ask if both the Eckart and Stewart ansatz's make sense or not. It is noteworthy that the most general derivation of the hydrodynamic equation on the basis of the phenomenological argument gives a class of equations which can allow the existence of the dissipative internal energy  $\delta e$  and the dissipative particle-number density  $\delta n$  as well as the standard dissipative pressure  $\delta p$ , as shown by the present authors;<sup>22)</sup> a brief recapitulation of Ref.22) is given in Appendix A.

(b) The unphysical instabilities of the equilibrium state might be attributed to the lack of causality, and Israel-Stewart's formalism is presently being examined in connection to this problem.<sup>8)-10),20)</sup> Although their equation may get rid of the instability problem with a choice of the relaxation times, as shown in Ref.17), we emphasize that there exists no connection between the unphysical instabilities and the lack of causality. In fact, the Landau-Lifshitz equation is free from the instabilities of the equilibrium state in contrast of the Eckart equation. Furthermore, one should notice that the causal equation by Israel and Stewart is an extended version of the Eckart equation and hence it can naturally exhibit unphysical instabilities depending on the values of transport coefficients and relaxation times contained in the equation.<sup>23)</sup>

## 1.2. *Strategy of the present work: separation of the scales of dynamics and construction of hydrodynamic equation as the infrared effective dynamics*

Here, we note the hierarchy of the dynamics of the time evolution of a many-body system: In the beginning of the time evolution of a prepared state, the whole dynamical evolution of the system will be governed by Hamiltonian dynamics that is time-reversal invariant. When the system becomes old, the dynamics is relaxed to the kinetic regime, where the time-evolution system is well described by a truncation of the BBGKY (Bogoliubov-Born-Green-Kirkwood-Yvon) hierarchy;<sup>24)</sup> the Boltzmann equation composed of one-body distribution function describes a coarse grained slower dynamics, in which time-reversal invariance is lost. Then, as the

system is further relaxed, the time evolution will be described in terms of the hydrodynamic quantities, i.e., the flow velocity, the particle-number density, and the local temperature. In this sense, the hydrodynamics is the infrared asymptotic dynamics of the kinetic equation.

Now the relativistic Boltzmann equation is a typical kinetic equation, which is manifestly Lorentz invariant and free from the instability and causal problems.<sup>18)</sup> Thus, one sees that a natural way to resolve the ambiguities in the definition of the flow velocity and the unphysical instabilities of the equilibrium state is to derive the relativistic dissipative hydrodynamic equation from an underlying relativistic kinetic equation. Indeed, there have been some vigorous attempts to derive the phenomenological equations from the relativistic Boltzmann equation; for instance, with use of the Chapman-Enskog expansion method<sup>25)</sup> and the Maxwell-Grad moment method.<sup>26)</sup> We would say, however, that these works in the microscopic approaches are not fully satisfactory: Although the past works certainly succeeded in identifying the assumptions and/or approximations to reproduce the known hydrodynamic equations by Eckart, Landau and Lifshitz, Stewart, and Israel, the physical meaning and foundation of these assumptions/approximations remain obscure, and thus the uniqueness of those hydrodynamic equations has never been elucidated as the long-wavelength and low-frequency limit of the underlying dynamics. Indeed, the standard derivation of relativistic hydrodynamic equations based on the Chapman-Enskog expansion or Maxwell-Grad moment method<sup>18)</sup> utilizes the ansatz's given by (i)  $\sim$  (v) as the constraints on the distribution function as the solution of the relativistic Boltzmann equation, rather than consequences of the derivation. Their validity or the fundamental compatibility with the underlying Boltzmann equation has never been questioned nor addressed. This unsatisfactory situation rather reveals the incompleteness of the Chapman-Enskog expansion method and the Maxwell-Grad moment methods themselves as a reduction theory of the dynamics.

Thus, the form of relativistic dissipative hydrodynamic equations is still controversial and far from being established. The origins of the difficulty of the derivation are identified as the absence of an appropriate coarse-graining method that keeps the Lorentz covariance and is applicable to the relativistic Boltzmann equation. It should be mentioned here that van Kampen<sup>16)</sup> applied his reduction theory to derive a relativistic hydrodynamic equation. The resultant equation was, unfortunately, of a noncovariant form.

In this work, we try to derive the relativistic dissipative hydrodynamic equation from the relativistic Boltzmann equation in a more natural and systematic way: For that, it is essential to adopt a powerful reduction theory of the dynamics.<sup>27)</sup> As such a reduction theory, we take the “renormalization-group (RG) method”. In Ref.'s 28)–35), it has been shown that the RG method is a powerful resummation method and also gives a systematic reduction theory of the dynamics leading to the coarse graining of temporal and spatial scales, which are the key concepts in the construction of infrared effective theories <sup>\*)</sup>. Indeed, the RG method is al-

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<sup>\*)</sup> A brief account of the RG method is given in Appendix B using an example for self-containedness.

ready applied satisfactorily to the derivation of the Navier-Stokes equation from the (non-relativistic) Boltzmann equation, with no heuristic assumption.<sup>34),35)</sup> The RG method thus should be most suitable for the present purpose to derive covariant relativistic dissipative hydrodynamic equations. It is also expected that the physical meanings and the validity of the ansatz's posed in the phenomenological derivation will be elucidated in the process of the reduction in the RG method.

In Ref.36), the present authors and K. Ohnishi applied the RG method to derive the relativistic dissipative hydrodynamic equations for the first time: A macroscopic-frame vector was introduced with which the derivation of a coarse-grained covariant equation is made possible, and thus the so-called *first-order* relativistic dissipative hydrodynamic equations were successfully derived, in which the dissipative effects are taken into account up to the first order. The five hydrodynamic variables naturally correspond to the zero modes of the linearized collision operator. The deviations from the local equilibrium are given by the functions that are precisely orthogonal to the zero modes with an inner product for the distribution functions. It was found that the various local rest frames of the flow velocity can be realized by a choice of the macroscopic-frame vector. Subsequently, the present authors<sup>37)</sup> showed that the derived hydrodynamic equation in the particle frame is different from the Eckart and Stewart equations, and has the stable equilibrium state in a numerical calculation for a rarefied gas where dissipative effects are most significant.

### 1.3. *Purpose of the present paper*

The purpose of this paper is as follows: (1) We present a detailed and full account of the derivation of the first-order relativistic dissipative hydrodynamic equations in generic frames from the relativistic Boltzmann equation on the basis of the RG method. Moreover, (2) we elaborate and revise some parts of the derivation and thereby make it more transparent: We clarify the essential importance to properly define the inner product of the distribution functions, in particular to make it positive-definite, which was not recognized in the previous presentation. We also show using simple properties of the inner product that the usually problematic Burnett term does not contribute to the hydrodynamic equations on account of the fact that the hydrodynamic modes are described by the zero modes of the linearized collision operator. (3) We show that the ansatz's (i)  $\sim$  (v) for the dissipative currents exactly correspond to the orthogonality conditions of the perturbed distribution function to the zeroth-order solution, and that the Eckart constraint (i)  $\sim$  (iii) can not be compatible with the underlying Boltzmann equation as a corollary. (4) We also fully examine the properties and its advantageous nature of the derived equation in a generic frame. We give a general proof without recourse to numerical calculations that the hydrodynamic equations obtained in our formalism have the stable equilibrium state even in the particle (Eckart) frame on the basis of the positive-definiteness of the inner product.

This paper is organized as follows: In §2, after a brief account of the basic ingredients of the relativistic Boltzmann equation, we introduce the macroscopic-frame vector, and summarize the ad-hoc aspects in the standard methods such as the Chapman-Enskog expansion and Maxwell-Grad moment methods. In §3, with

use of the RG method, we reduce the relativistic Boltzmann equation to a generic form of the relativistic dissipative hydrodynamic equation, whose frame is not specified. In §4, we show that the obtained equation reduces to the relativistic dissipative hydrodynamic equations in various frames with a choice of the macroscopic-frame vector, including the particle one and the energy one. Then, we compare our equations with those proposed by Eckart, Landau-Lifshitz, and Stewart. In §5.1 and §5.2, we examine some properties of our equations, concerning transport coefficients and frames. In §5.3, we present a proof that all of our equations have the stable equilibrium state. The last section is devoted to a summary and concluding remarks. In Appendix A, we show that the most general form of the hydrodynamic equation as derived in the standard phenomenological way may admit the dissipative internal energy, pressure, and particle-number density. In Appendix B, after giving a brief account of the RG method using an example, we present a general ground of this method. In Appendix C, we present a detailed derivation of the first-excited modes in a generic local rest frame.

## §2. Preliminaries

After a brief account of the basic properties of the relativistic Boltzmann equation, we introduce a macroscopic-frame vector to specify a local rest frame on which the macroscopic dynamics is described: We shall see that the introduction of the macroscopic-frame vector enables us to have a coarse-grained and covariant equation. As in the standard method like the Chapman-Enskog expansion method and others,<sup>18)</sup> we take the spatial inhomogeneity as the origin of the dissipation. We shall clarify the ad-hoc aspects of the ansatz's made in the standard Chapman-Enskog expansion and Maxwell-Grad moment methods for the derivation of the relativistic hydrodynamic equations.

### 2.1. Relativistic Boltzmann equation

As a simple example, we shall treat the classical relativistic system composed of identical particles<sup>\*)</sup>. Then, the relativistic Boltzmann equation<sup>18)</sup> for such a system reads

$$p^\mu \partial_\mu f_p(x) = C[f]_p(x), \quad (2.1)$$

where  $f_p(x)$  denotes the one-particle distribution function defined in the phase space  $(x, p)$  with  $x^\mu$  being the space-time coordinate and  $p^\mu$  being the four momentum of the on-shell particle;  $p^\mu p_\mu = p^2 = m^2$  and  $p^0 > 0$ . The right-hand side of Eq.(2.1) is called the collision integral,

$$C[f]_p(x) \equiv \frac{1}{2!} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p, p_1 | p_2, p_3) \left( f_{p_2}(x) f_{p_3}(x) - f_p(x) f_{p_1}(x) \right), \quad (2.2)$$

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<sup>\*)</sup> An extension to multi-component systems is possible and will be presented elsewhere.

where  $\omega(p, p_1|p_2, p_3)$  denotes the transition probability due to the microscopic two-particle interaction. To make explicit the correspondence to the general formulation given in Ref.33), we treat the momentum as a discrete variable, but the summation with respect to the momentum is interpreted as the integration;  $\sum_q \equiv \int d^3\mathbf{q}$  with  $\mathbf{q}$  being the spatial components of the four momentum  $q^\mu$ . We remark that the transition probability  $\omega(p, p_1|p_2, p_3)$  contains the delta functions representing the energy-momentum conservation,

$$\omega(p, p_1|p_2, p_3) \propto \delta^4(p + p_1 - p_2 - p_3), \quad (2.3)$$

and also has the symmetric properties due to the indistinguishability of the particles and the time reversal invariance of the microscopic transition probability,

$$\omega(p, p_1|p_2, p_3) = \omega(p_2, p_3|p, p_1) = \omega(p_1, p|p_3, p_2) = \omega(p_3, p_2|p_1, p). \quad (2.4)$$

It should be stressed here that we have confined ourselves to the case in which the particle number is conserved in the collision process.

The property of the transition probability shown in Eq.(2.4) leads to the following identity satisfied for an arbitrary vector  $\varphi_p(x)$ ,

$$\begin{aligned} \sum_p \frac{1}{p^0} \varphi_p(x) C[f]_p(x) &= \frac{1}{2!} \frac{1}{4} \sum_p \frac{1}{p^0} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \\ &\times \omega(p, p_1|p_2, p_3) \left( \varphi_p(x) + \varphi_{p_1}(x) - \varphi_{p_2}(x) - \varphi_{p_3}(x) \right) \\ &\times \left( f_{p_2}(x) f_{p_3}(x) - f_p(x) f_{p_1}(x) \right). \end{aligned} \quad (2.5)$$

A function  $\varphi_p(x)$  is called a collision invariant (or summational invariant<sup>18)</sup>) when it satisfies the following equation

$$\sum_p \frac{1}{p^0} \varphi_p(x) C[f]_p(x) = 0. \quad (2.6)$$

As is easily confirmed by using the formula (2.5) and the property (2.3),  $\varphi_p = 1$  and  $p^\mu$  are collision invariants;

$$\sum_p \frac{1}{p^0} C[f]_p(x) = 0, \quad (2.7)$$

$$\sum_p \frac{1}{p^0} p^\mu C[f]_p(x) = 0, \quad (2.8)$$

which represent, of course, the conservation of the particle number, energy, and momentum by the collision process, respectively. We also see that the linear combination of these collision invariants as given by

$$\varphi_p(x) = a(x) + p^\mu b_\mu(x), \quad (2.9)$$

is also a collision invariant with  $a(x)$  and  $b_\mu(x)$  being arbitrary functions of  $x$ . This form is, in fact, known to be the most general form of a collision invariant.<sup>18)</sup>

On account of Eq.'s (2·7) and (2·8), we have the continuity or balance equations for the particle current  $N^\mu$  and the energy-momentum tensor  $T^{\mu\nu}$ ,

$$\partial_\mu N^\mu(x) \equiv \partial_\mu \left[ \sum_p \frac{1}{p^0} p^\mu f_p(x) \right] = 0, \quad (2\cdot10)$$

$$\partial_\nu T^{\mu\nu}(x) \equiv \partial_\nu \left[ \sum_p \frac{1}{p^0} p^\mu p^\nu f_p(x) \right] = 0, \quad (2\cdot11)$$

respectively. It is noted that while these equations have the same forms as the hydrodynamic equations, nothing about the dynamical properties is contained in these equations before the evolution of the distribution function  $f_p(x)$  is obtained by solving Eq.(2·1). In the standard Chapman-Enskog expansion method,<sup>18)</sup> these balance equations are rather used to obtain the time derivatives of the distribution function written in terms of the hydrodynamic quantities, order by order.

The entropy current is defined by

$$S^\mu(x) \equiv - \sum_p \frac{1}{p^0} p^\mu f_p(x) (\ln f_p(x) - 1). \quad (2\cdot12)$$

Using the relativistic Boltzmann equation (2·1), the divergence of the entropy current reads

$$\partial_\mu S^\mu(x) = - \sum_p \frac{1}{p^0} C[f]_p(x) \ln f_p(x). \quad (2\cdot13)$$

The above equation tells us that  $S^\mu$  is conserved only if  $\ln f_p(x)$  is a collision invariant, or a linear combination of the basic collision invariants  $(1, p^\mu)$  as

$$\ln f_p(x) = \alpha(x) + p^\mu \beta_\mu(x), \quad (2\cdot14)$$

with  $\alpha(x)$  and  $\beta_\mu(x)$  being arbitrary functions of  $x$ . In other words, the entropy-conserving distribution function is parametrized as

$$f_p(x) = \frac{1}{(2\pi)^3} \exp \left[ \frac{\mu(x) - p^\mu u_\mu(x)}{T(x)} \right] \equiv f_p^{\text{eq}}(x), \quad (2\cdot15)$$

with  $u^\mu(x) u_\mu(x) = 1$ , which is identified with the local equilibrium distribution function called the Jüttner function<sup>38)</sup> (the relativistic analog of the Maxwellian):  $T(x)$ ,  $\mu(x)$ , and  $u^\mu(x)$  in Eq.(2·15) should be interpreted as the local temperature, the chemical potential, and the flow velocity, respectively.

We note that for the local equilibrium distribution  $f_p^{\text{eq}}(x)$  the collision integral identically vanishes,

$$C[f^{\text{eq}}]_p(x) = 0, \quad (2\cdot16)$$

due to the energy-momentum conservation implemented in the transition probability (2·3).



## 2.2. Introduction of macroscopic-frame vector

We are now in a position to introduce the key ingredient in the present work.

Since we are interested in the hydrodynamic regime where the time and space dependence of the physical quantities are small, we try to solve Eq.(2.1) in the hydrodynamic regime where the space-time variation of  $f_p(x)$  is small and the space-time scales are coarse-grained from those in the kinetic regime. To make a coarse graining with the Lorentz covariance being retained, we introduce a time-like Lorentz vector denoted by

$$\mathbf{a}^\mu, \quad (2.17)$$

with

$$\mathbf{a}^0 > 0. \quad (2.18)$$

Thus,  $\mathbf{a}^\mu$  specify the covariant but macroscopic coordinate system where the velocity field of the hydrodynamic flow is defined: Since such a coordinate system is called *frame*, we call  $\mathbf{a}^\mu$  the *macroscopic-frame vector*. Although  $\mathbf{a}^\mu$  may depend on the momentum  $p$  and the space-time coordinate  $x$ , i.e.,

$$\mathbf{a}^\mu = \mathbf{a}_p^\mu(x), \quad (2.19)$$

the time variation of it is supposed to be much smaller than that of the microscopic processes. We shall see that the separation of the scales between the kinetic and hydrodynamic regimes can be nicely achieved by the RG method. It should be stressed here that although a macroscopic vector is introduced also in the standard Chapman-Enskog expansion method,<sup>18)</sup> the vector is identified as the flow velocity  $u^\mu(x)$  from the outset. In our case,  $\mathbf{a}_p^\mu(x)$  does not necessarily coincide with the flow velocity  $u^\mu(x)$ .

Keeping in mind the above scale difference, we decompose the derivative  $\partial^\mu$  into time-like and space-like ones in terms of the macroscopic-frame vector. Defining a projection operator to the space-like vector by

$$\Delta_p^{\mu\nu}(x) \equiv g^{\mu\nu} - \frac{\mathbf{a}_p^\mu(x) \mathbf{a}_p^\nu(x)}{\mathbf{a}_p^2(x)}, \quad (2.20)$$

we have

$$\partial^\mu = \frac{\mathbf{a}_p^\mu(x) \mathbf{a}_p^\nu(x)}{\mathbf{a}_p^2(x)} \partial_\nu + \Delta_p^{\mu\nu}(x) \partial_\nu = \mathbf{a}_p^\mu(x) \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \sigma_\mu}, \quad (2.21)$$

with

$$\frac{\partial}{\partial \tau} \equiv \frac{1}{\mathbf{a}_p^2(x)} \mathbf{a}_p^\nu(x) \partial_\nu, \quad (2.22)$$

$$\frac{\partial}{\partial \sigma_\mu} \equiv \Delta_p^{\mu\nu}(x) \partial_\nu \equiv \nabla^\mu. \quad (2.23)$$

Then, the relativistic Boltzmann equation (2.1) in the new coordinate system  $(\tau, \sigma^\mu)$  is written as

$$p \cdot \mathbf{a}_p(\tau, \sigma) \frac{\partial}{\partial \tau} f_p(\tau, \sigma) + p \cdot \nabla f_p(\tau, \sigma) = C[f]_p(\tau, \sigma), \quad (2.24)$$

where  $\mathbf{a}_p^\mu(\tau, \sigma) \equiv \mathbf{a}_p^\mu(x)$  and  $f_p(\tau, \sigma) \equiv f_p(x)$ . We remark the prefactor of the time derivative is a Lorentz scalar and positive definite;

$$p \cdot \mathbf{a}_p(\tau, \sigma) > 0, \quad (2.25)$$

which is easily verified by taking the rest frame of  $p^0$ .

Since we are interested in a hydrodynamic solution to Eq.(2.24) as mentioned above, we shall convert Eq.(2.24) into

$$\frac{\partial}{\partial \tau} f_p(\tau, \sigma) = \frac{1}{p \cdot \mathbf{a}_p(\tau, \sigma)} C[f]_p(\tau, \sigma) - \varepsilon \frac{1}{p \cdot \mathbf{a}_p(\tau, \sigma)} p \cdot \nabla f_p(\tau, \sigma), \quad (2.26)$$

where a small quantity  $\varepsilon$  has been introduced to express that the space derivatives are small for the system which we are interested in;  $\varepsilon$  is called the non-uniformity parameter<sup>18)</sup> and may be identified with the ratio of the average particle distance over the mean free path, i.e., the Knudsen number.

Since  $\varepsilon$  appears in front of the second term of the right-hand side of Eq.(2.26), the relativistic Boltzmann equation has a form to which the perturbative expansion is applicable. In fact, this form of the Boltzmann equation but with  $\mathbf{a}_p^\mu = u^\mu$  is also the starting point for the standard Chapman-Enskog expansion method,<sup>18)</sup> together with an ad-hoc ansatz on the order of the time derivative of the distribution function. We stress that this seemingly mere rewrite of the equation has a physical significance; it expresses a natural assumption that only the spatial inhomogeneity over distances of the order of the mean free path is the origin of the dissipation. We remark that the RG method applied to non-relativistic Boltzmann equation with the corresponding assumption leads to the Navier-Stokes equation;<sup>25), 27), 34), 35)</sup> the present approach is simply a covariantization of the non-relativistic case.

For setting up the perturbative expansion in a consistent way with the physical picture of the origin of the dissipation, we shall take the coordinate system where  $\mathbf{a}_p^\mu(\tau, \sigma)$  has no  $\tau$  dependence, i.e.,

$$\mathbf{a}_p^\mu(\tau, \sigma) = \mathbf{a}_p^\mu(\sigma). \quad (2.27)$$

Then, Eq.(2.26) takes the following form,

$$\frac{\partial}{\partial \tau} f_p(\tau, \sigma) = \frac{1}{p \cdot \mathbf{a}_p(\sigma)} C[f]_p(\tau, \sigma) - \varepsilon \frac{1}{p \cdot \mathbf{a}_p(\sigma)} p \cdot \nabla f_p(\tau, \sigma). \quad (2.28)$$

Now let  $q_p(\sigma)$  is a physical quantity of a particle with a momentum  $p^\mu$  at a position  $\sigma^\mu$ ; then the total amount of the quantity is given by

$$Q_p(\tau) = \int d^3\sigma q_p(\sigma) f_p(\tau, \sigma). \quad (2.29)$$

When  $q_p(\sigma) = p \cdot \mathbf{a}_p(\sigma)$ , the time variation of  $Q_p(\tau)$  is given by

$$\begin{aligned} \frac{d}{d\tau} Q_p(\tau) &= \int d^3\sigma p \cdot \mathbf{a}_p(\sigma) \frac{\partial}{\partial \tau} f_p(\tau, \sigma), \\ &= -\varepsilon p^\mu \int d^3\sigma \frac{\partial}{\partial \sigma^\mu} f_p(\tau, \sigma) + \int d^3\sigma C[f]_p(\tau, \sigma), \\ &= \int d^3\sigma C[f]_p(\tau, \sigma), \end{aligned} \quad (2.30)$$

where we have used Eq.(2.28) and neglected the contribution from the spatial boundary of the system. Here, we consider a trajectory of one particle in the interval between each collisions: Substituting  $C[f]_p(\tau, \sigma) = 0$  into Eq.(2.30), we find that  $Q_p(\tau)$  is a conserved quantity and  $q_p(\sigma)$  is a corresponding density. Thus, the form of the relativistic Boltzmann equation as given in Eq.(2.28) tells us that the physical quantity that is transported by each particle in the system is given by

$$p \cdot \mathbf{a}_p(\sigma). \quad (2.31)$$

It is to be noted that we can control what is the flow represented in our theory by varying the specific expression of  $\mathbf{a}_p^\mu(\sigma)$ . Because of this freedom inherent in our coordinate system, our theory may lead to various hydrodynamic equations, including the ones in the energy and particle frames for non-ideal fluids.

### 2.3. A brief description of the standard methods

Before developing our analysis based on the RG method, we briefly summarize the ad-hoc aspects in the standard methods such as the Chapman-Enskog expansion and Maxwell-Grad moment methods.

In the standard Chapman-Enskog expansion method, one starts from the following form of the Boltzmann equation,

$$p \cdot u D_{\text{CE}} f_p(x) = -\varepsilon p \cdot \nabla_{\text{CE}} f_p(x) + C[f]_p(x), \quad (2.32)$$

where  $D_{\text{CE}} \equiv u^\mu \partial_\mu$  and  $\nabla_{\text{CE}}^\mu \equiv \Delta_{\text{CE}}^{\mu\nu} \partial_\nu$  with  $\Delta_{\text{CE}}^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ . Then, one makes the perturbative expansion

$$f_p(x) = f_p^{(0)}(x) + \varepsilon f_p^{(1)}(x) + \varepsilon^2 f_p^{(2)}(x) + \cdots \equiv f_p^{(0)}(x) + \delta f_p(x), \quad (2.33)$$

$$D_{\text{CE}} f_p(x) = \varepsilon D_{\text{CE}} f_p^{(1)}(x) + \varepsilon^2 D_{\text{CE}} f_p^{(2)}(x) + \cdots. \quad (2.34)$$

Then, it is found that the zeroth-order solution is given by the local equilibrium distribution function, i.e., the Jüttner function given by Eq.(2.15);

$$f_p^{(0)}(x) = f_p^{\text{eq}}(x). \quad (2.35)$$

It is customary,<sup>18)</sup> without any physical foundation, to assume that the particle-number density and internal energy in the non-equilibrium state is the same as those in the local equilibrium state, and set as follows;

$$n \equiv u_\mu \left[ \sum_p \frac{1}{p^0} p^\mu f_p \right] = u_\mu \left[ \sum_p \frac{1}{p^0} p^\mu f_p^{(0)} \right], \quad (2.36)$$

$$e \equiv u_\mu \left[ \sum_p \frac{1}{p^0} p^\mu p^\nu f_p \right] u_\nu = u_\mu \left[ \sum_p \frac{1}{p^0} p^\mu p^\nu f_p^{(0)} \right] u_\nu. \quad (2.37)$$

For consistency, one also imposes the constraints to the higher-order terms

$$\delta n = u_\mu \left[ \sum_p \frac{1}{p^0} p^\mu \delta f_p \right] = 0, \quad (2.38)$$

$$\delta e = u_\mu \left[ \sum_p \frac{1}{p^0} p^\mu p^\nu \delta f_p \right] u_\nu = 0. \quad (2.39)$$

To obtain the hydrodynamic equation in the particle (Eckart) frame, another constraint is imposed

$$\nu^\mu = \Delta_{\text{CE}}^{\mu\nu} \delta N_\nu = \Delta_{\text{CE}}^{\mu\nu} \left[ \sum_p \frac{1}{p^0} p_\nu \delta f_p \right] = 0. \quad (2.40)$$

Instead, if one wants to obtain the hydrodynamic equation in the energy (Landau-Lifshitz) frame, one also imposes the constraint

$$Q^\mu = \Delta_{\text{CE}}^{\mu\nu} \delta T_{\nu\rho} u^\rho = \Delta_{\text{CE}}^{\mu\nu} \left[ \sum_p \frac{1}{p^0} p_\nu p_\rho \delta f_p \right] u^\rho = 0. \quad (2.41)$$

The constraints imposed to the distribution function in the higher orders are called the *conditions of fit*; the zeroth-order constraint is not a constraint but an identity.

It is noteworthy that although a foundation for them has never been given, such conditions of fit (2.36)-(2.41) are also imposed in an ad-hoc way even when the Maxwell-Grad moment method is adopted.<sup>18)</sup> We stress here that these constraints are actually equivalent with a strong physical assumption that there are no particle-number density nor internal energy of the dissipative origin, although the distribution function in the non-equilibrium state is quite different from that in the local equilibrium state. It is, therefore, an urgent but yet unsolved problem to verify or elaborate these ad-hoc constraints somehow, say, from a microscopic theory, or by experiment, if possible.

In the RG method which we adopt, one needs no such conditions of fit for derivation, and rather the correct forms of them are obtained as a property of the derived equation once the frame is specified by the macroscopic-frame vector: We shall see that the conditions of fit in the energy frame is compatible with the underlying Boltzmann equation and physical, but those in the particle frame is not and thus will never be satisfied in any physical system.

### §3. Reduction of Relativistic Boltzmann Equation with Renormalization-group Method

In this section, starting from Eq.(2.28) we shall derive the relativistic dissipative hydrodynamic equation as the infrared asymptotic dynamics of the classical relativistic Boltzmann equation on the basis of the RG method: The five hydrodynamic

variables, i.e., the flow velocity, local temperature, and particle-number density (or chemical potential), naturally correspond to the zero modes of the linearized collision operator. Then, without recourse to any ansatz such as the conditions of fit given by Eq.'s (2·36)-(2·41), the excited modes which are to modify the local equilibrium distribution function are naturally defined in the sense that they are precisely orthogonal to the zero modes with a properly defined inner product for the distribution functions. It will be shown on the basis of the inner product that the so-called Burnett term does not affect the hydrodynamic equation owing to the fact that the hydrodynamic modes are the zero modes of the linearized collision operator.

### 3.1. Construction of the approximate solution around arbitrary initial time

In accordance with the general formulation of the RG method<sup>(29),(30),(32),(33)</sup> \*), we first try to obtain the perturbative solution  $\tilde{f}_p$  to Eq.(2·28) around the arbitrary initial time  $\tau = \tau_0$  with the initial value  $f_p(\tau_0, \sigma)$ ;

$$\tilde{f}_p(\tau = \tau_0, \sigma; \tau_0) = f_p(\tau_0, \sigma), \quad (3.1)$$

where we have made explicit that the solution has the  $\tau_0$  dependence. The initial value is not yet specified, we suppose that the initial value is on an exact solution. The initial value as well as the solution is expanded with respect to  $\varepsilon$  as follows;

$$\tilde{f}_p(\tau, \sigma; \tau_0) = \tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) + \varepsilon \tilde{f}_p^{(1)}(\tau, \sigma; \tau_0) + \varepsilon^2 \tilde{f}_p^{(2)}(\tau, \sigma; \tau_0) + \cdots, \quad (3.2)$$

and

$$f_p(\tau_0, \sigma) = f_p^{(0)}(\tau_0, \sigma) + \varepsilon f_p^{(1)}(\tau_0, \sigma) + \varepsilon^2 f_p^{(2)}(\tau_0, \sigma) + \cdots. \quad (3.3)$$

The respective initial conditions at  $\tau = \tau_0$  are set up as

$$\tilde{f}_p^{(l)}(\tau_0, \sigma; \tau_0) = f_p^{(l)}(\tau_0, \sigma) \text{ for } l = 0, 1, 2, \cdots. \quad (3.4)$$

In the expansion, the zeroth-order value  $\tilde{f}_p^{(0)}(\tau_0, \sigma; \tau_0) = f_p^{(0)}(\tau_0, \sigma)$  is supposed to be as close as possible to an exact solution.

Substituting the above expansions into Eq.(2·28) in the  $\tau$ -independent but  $\tau_0$ -dependent coordinate system with

$$\mathbf{a}_p^\mu(\tau, \sigma) = \mathbf{a}_p^\mu(\tau_0, \sigma) \equiv \mathbf{a}_p^\mu(\sigma; \tau_0), \quad (3.5)$$

we obtain the series of the perturbative equations with respect to  $\varepsilon$ .

Now the zeroth-order equation reads

$$\frac{\partial}{\partial \tau} \tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) = \frac{1}{p \cdot \mathbf{a}_p(\sigma; \tau_0)} C[\tilde{f}_p^{(0)}]_p(\tau, \sigma; \tau_0). \quad (3.6)$$

Since we are interested in the slow motion which would be realized asymptotically as  $\tau \rightarrow \infty$ , we should take the following stationary solution or the fixed point,

$$\frac{\partial}{\partial \tau} \tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) = 0, \quad (3.7)$$

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\*) See Appendix B for a brief but self-contained account of the RG method.

which is realized when

$$\frac{1}{p \cdot \mathbf{a}_p(\sigma; \tau_0)} C[\tilde{f}^{(0)}]_p(\tau, \sigma; \tau_0) = 0, \quad (3.8)$$

for arbitrary  $\sigma$ . Thus, we see that  $\ln \tilde{f}_p^{(0)}(\tau, \sigma; \tau_0)$  can be represented as a linear combination of the five collision invariants  $(1, p^\mu)$  as mentioned in the last section, and hence  $\tilde{f}_p^{(0)}(\tau, \sigma; \tau_0)$  is found to be a local equilibrium distribution function and thus given by the Jüttner function (2.15):

$$\tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) = \frac{1}{(2\pi)^3} \exp \left[ \frac{\mu(\sigma; \tau_0) - p^\mu u_\mu(\sigma; \tau_0)}{T(\sigma; \tau_0)} \right] \equiv f_p^{\text{eq}}(\sigma; \tau_0). \quad (3.9)$$

with  $u^\mu(\sigma; \tau_0) u_\mu(\sigma; \tau_0) = 1$ , which implies that

$$f_p^{(0)}(\tau_0, \sigma) = \tilde{f}_p^{(0)}(\tau_0, \sigma; \tau_0) = f_p^{\text{eq}}(\sigma; \tau_0). \quad (3.10)$$

It should be noticed that the five would-be integration constants  $T(\sigma; \tau_0)$ ,  $\mu(\sigma; \tau_0)$ , and  $u_\mu(\sigma; \tau_0)$  are independent of  $\tau$  but may depend on  $\tau_0$  as well as  $\sigma$ . In the following, we shall suppress the coordinate arguments  $(\sigma; \tau_0)$  and the momentum subscript, e.g.,  $p$  when no misunderstanding is expected.

### 3.2. The linearized collision operator and inner product

Now the first-order equation reads

$$\frac{\partial}{\partial \tau} \tilde{f}_p^{(1)}(\tau) = \sum_q A_{pq} \tilde{f}_q^{(1)}(\tau) + F_p, \quad (3.11)$$

where the linear evolution operator  $A$  and the inhomogeneous term  $F$  are defined by

$$\begin{aligned} A_{pq} &\equiv \frac{1}{p \cdot \mathbf{a}_p} \frac{\partial}{\partial f_q} C[f]_p \Big|_{f=f^{\text{eq}}} \\ &= \frac{1}{p \cdot \mathbf{a}_p} \frac{1}{2!} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p, p_1 | p_2, p_3) \\ &\quad \times (\delta_{p_2q} f_{p_3}^{\text{eq}} + f_{p_2}^{\text{eq}} \delta_{p_3q} - \delta_{pq} f_{p_1}^{\text{eq}} - f_p^{\text{eq}} \delta_{p_1q}), \end{aligned} \quad (3.12)$$

and

$$F_p \equiv -\frac{1}{p \cdot \mathbf{a}_p} p \cdot \nabla f_p^{\text{eq}}, \quad (3.13)$$

respectively.

To obtain the solution which describes a slow motion, it is convenient to first analyze the spectral properties of  $A$ . For this purpose, we convert  $A$  to another linear operator

$$L \equiv f^{\text{eq}-1} A f^{\text{eq}}, \quad (3.14)$$

with the diagonal matrix  $f_{pq}^{\text{eq}} \equiv f_p^{\text{eq}} \delta_{pq}$ ; the explicit form of  $L$  is given by

$$L_{pq} = \frac{-1}{p \cdot \mathbf{a}_p} \frac{1}{2!} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p, p_1 | p_2, p_3) f_{p_1}^{\text{eq}} (\delta_{pq} + \delta_{p_1 q} - \delta_{p_2 q} - \delta_{p_3 q}). \quad (3.15)$$

Here, we have used the identity

$$\omega(p, p_1 | p_2, p_3) f_{p_2}^{\text{eq}} f_{p_3}^{\text{eq}} = \omega(p, p_1 | p_2, p_3) f_p^{\text{eq}} f_{p_1}^{\text{eq}}, \quad (3.16)$$

which follows from Eq.'s (2.3) and (3.9).

Let us define the inner product between arbitrary non-zero vectors  $\varphi$  and  $\psi$  by

$$\langle \varphi, \psi \rangle \equiv \sum_p \frac{1}{p^0} (p \cdot \mathbf{a}_p) f_p^{\text{eq}} \varphi_p \psi_p. \quad (3.17)$$

We note that the norm defined through this inner product is positive definite

$$\langle \varphi, \varphi \rangle = \sum_p \frac{1}{p^0} (p \cdot \mathbf{a}_p) f_p^{\text{eq}} (\varphi_p)^2 > 0 \text{ for } \varphi_p \neq 0, \quad (3.18)$$

since

$$p \cdot \mathbf{a}_p > 0 \quad (3.19)$$

in accord with Eq.(2.25). Notice that the other factors than  $p \cdot \mathbf{a}_p$  in Eq.(3.18) are all positive definite. We shall see that this positive definiteness (3.18) of the inner product plays an essential role in making the resultant hydrodynamic equations assure the stability of the thermal equilibrium state, as it should be, in contrast to some phenomenological equations.

With the inner product, it is found that  $L$  is self-adjoint

$$\begin{aligned} \langle \varphi, L \psi \rangle &= -\frac{1}{2!} \frac{1}{4} \sum_p \frac{1}{p^0} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \\ &\quad \omega(p, p_1 | p_2, p_3) f_p^{\text{eq}} f_{p_1}^{\text{eq}} (\varphi_p + \varphi_{p_1} - \varphi_{p_2} - \varphi_{p_3}) (\psi_p + \psi_{p_1} - \psi_{p_2} - \psi_{p_3}) \\ &= \langle L \varphi, \psi \rangle, \end{aligned} \quad (3.20)$$

and semi-negative definite

$$\begin{aligned} \langle \varphi, L \varphi \rangle &= -\frac{1}{2!} \frac{1}{4} \sum_p \frac{1}{p^0} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \\ &\quad \times \omega(p, p_1 | p_2, p_3) f_p^{\text{eq}} f_{p_1}^{\text{eq}} (\varphi_p + \varphi_{p_1} - \varphi_{p_2} - \varphi_{p_3})^2 \\ &\leq 0, \end{aligned} \quad (3.21)$$

which means that the eigen values of  $L$  are zero or negative. In the derivation of Eq's (3.20) and (3.21), we have used Eq.'s (2.3) and (3.16).

The eigen vectors belonging to the zero eigen value are found to be

$$\varphi_{0p}^\alpha \equiv \begin{cases} p^\mu & \text{for } \alpha = \mu, \\ 1 \times m & \text{for } \alpha = 4, \end{cases} \quad (3.22)$$

which span the kernel of  $L$  and satisfy

$$[L \varphi_0^\alpha]_p = 0. \quad (3.23)$$

We call  $\varphi_0^\alpha$  the *zero modes*. It is noted that these zero modes described by the five vectors are collision invariants shown in Eq.'s (2.7) and (2.8), and the factor  $m$  in  $\varphi_{0p}^4$  is introduced merely for convenience so that our method can be applied to the case of massless particles.

Following Ref.33), we define the projection operator  $P_0$  onto the kernel of  $L$  which is called the  $P_0$  space and the projection operator  $Q_0$  onto the  $Q_0$  space complement to the  $P_0$  space:

$$[P_0 \psi]_p \equiv \varphi_{0p}^\alpha \eta_{0\alpha\beta}^{-1} \langle \varphi_0^\beta, \psi \rangle, \quad (3.24)$$

$$Q_0 \equiv 1 - P_0, \quad (3.25)$$

where  $\eta_{0\alpha\beta}^{-1}$  is the inverse matrix of the the  $P_0$ -space metric matrix  $\eta_0^{\alpha\beta}$  defined by

$$\eta_0^{\alpha\beta} \equiv \langle \varphi_0^\alpha, \varphi_0^\beta \rangle. \quad (3.26)$$

### 3.3. First-order solution

The solution to Eq.(3.11) with the initial condition  $\tilde{f}^{(1)}(\tau = \tau_0) = f^{(1)}$ , i.e.,  $\tilde{f}_p^{(1)}(\tau = \tau_0, \sigma; \tau_0) = f_p^{(1)}(\sigma; \tau_0)$  is expressed as

$$\tilde{f}^{(1)}(\tau) = e^{(\tau - \tau_0)A} \left\{ f^{(1)} + A^{-1} \bar{Q}_0 F \right\} + (\tau - \tau_0) \bar{P}_0 F - A^{-1} \bar{Q}_0 F, \quad (3.27)$$

where we have introduced the modified projection operators

$$\bar{P}_0 \equiv f^{\text{eq}} P_0 f^{\text{eq}-1}, \quad (3.28)$$

$$\bar{Q}_0 \equiv f^{\text{eq}} Q_0 f^{\text{eq}-1}. \quad (3.29)$$

We remark that the first term in Eq.(3.27) would be a fast motion coming from the  $Q_0$  space, which can be simply eliminated by choosing the initial value  $f^{(1)}$ , which has not yet been specified, as

$$f^{(1)} = \tilde{f}^{(1)}(\tau_0) = -A^{-1} \bar{Q}_0 F. \quad (3.30)$$

Thus, we have the first-order solution

$$\tilde{f}^{(1)}(\tau) = (\tau - \tau_0) \bar{P}_0 F - A^{-1} \bar{Q}_0 F, \quad (3.31)$$

with the initial value (3.30). We notice the appearance of the secular term proportional to  $\tau - \tau_0$ , which apparently invalidates the perturbative solution when  $|\tau - \tau_0|$



becomes large. It is worth mentioning that the standard Chapman-Enskog expansion method includes a set of conditions for not making secular terms appear; the conditions are the solvability conditions of the balance equations (2·10) and (2·11).<sup>18)</sup> In applying the solubility conditions, one needs apply the ad-hoc constraints on the distribution function for defining the flow (eg. Eckart flow or Landau-Lifshitz flow) as well as the ad-hoc constraints on the particle-number density and internal energy,<sup>18)</sup> as given by Eq.'s(2·36)-(2·41). In the present RG method, secular terms are allowed to appear and no constraints are imposed on the distribution function; rather the secular terms will be utilized to obtain the slow dynamics.

We remark here that we could apply the RG method here to Eq.(3·31), which will give the relativistic Euler equation without dissipation effects. To get a dissipative hydrodynamic equation, we need to proceed to the second order in our method.

### 3.4. Second-order solution

The second-order equation is written as

$$\frac{\partial}{\partial \tau} \tilde{f}_p^{(2)}(\tau) = \sum_q A_{pq} \tilde{f}_q^{(2)}(\tau) + (\tau - \tau_0)^2 G_p + (\tau - \tau_0) H_p + I_p, \quad (3·32)$$

where

$$\begin{aligned} B_{pqr} &\equiv \frac{1}{p \cdot \mathbf{a}_p} \frac{1}{2} \frac{\partial^2}{\partial f_q \partial f_r} C[f]_p \Big|_{f=f^{\text{eq}}} \\ &= \frac{1}{p \cdot \mathbf{a}_p} \frac{1}{2} \frac{1}{2!} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p, p_1 | p_2, p_3) \\ &\quad \times (\delta_{p_2q} \delta_{p_3r} + \delta_{p_2r} \delta_{p_3q} - \delta_{pq} \delta_{p_1r} - \delta_{pr} \delta_{p_1q}), \end{aligned} \quad (3·33)$$

$$G_p \equiv \sum_q \sum_r B_{pqr} [\bar{P}_0 F]_q [\bar{P}_0 F]_r, \quad (3·34)$$

$$\begin{aligned} H_p &\equiv - \sum_q \sum_r B_{pqr} \left( [\bar{P}_0 F]_q [A^{-1} \bar{Q}_0 F]_r + [A^{-1} \bar{Q}_0 F]_q [\bar{P}_0 F]_r \right) \\ &\quad - \frac{1}{p \cdot \mathbf{a}_p} p \cdot \nabla [\bar{P}_0 F]_p, \end{aligned} \quad (3·35)$$

$$I_p \equiv \sum_q \sum_r B_{pqr} [A^{-1} \bar{Q}_0 F]_q [A^{-1} \bar{Q}_0 F]_r + \frac{1}{p \cdot \mathbf{a}_p} p \cdot \nabla [A^{-1} \bar{Q}_0 F]_p. \quad (3·36)$$

The solution to Eq.(3·32) is found to be

$$\begin{aligned} \tilde{f}^{(2)}(\tau) &= e^{(\tau - \tau_0)A} \left\{ f^{(2)} + 2 A^{-3} \bar{Q}_0 G + A^{-2} \bar{Q}_0 H + A^{-1} \bar{Q}_0 I \right\} \\ &\quad + \frac{1}{3} (\tau - \tau_0)^3 \bar{P}_0 G + \frac{1}{2} (\tau - \tau_0)^2 \left\{ \bar{P}_0 H - 2 A^{-1} \bar{Q}_0 G \right\} \\ &\quad + (\tau - \tau_0) \left\{ \bar{P}_0 I - 2 A^{-2} \bar{Q}_0 G - A^{-1} \bar{Q}_0 H \right\} \\ &\quad + \left\{ - 2 A^{-3} \bar{Q}_0 G - A^{-2} \bar{Q}_0 H - A^{-1} \bar{Q}_0 I \right\}. \end{aligned} \quad (3·37)$$

Again the would-be fast motion can be eliminated by a choice of the initial value  $f^{(2)}$  so that

$$f^{(2)} = \tilde{f}^{(2)}(\tau_0) = -2A^{-3}\bar{Q}_0 G - A^{-2}\bar{Q}_0 H - A^{-1}\bar{Q}_0 I. \quad (3.38)$$

Then, we have a second-order solution

$$\begin{aligned} \tilde{f}^{(2)}(\tau) = & \frac{1}{3}(\tau - \tau_0)^3 \bar{P}_0 G + \frac{1}{2}(\tau - \tau_0)^2 \left\{ \bar{P}_0 H - 2A^{-1}\bar{Q}_0 G \right\} \\ & + (\tau - \tau_0) \left\{ \bar{P}_0 I - 2A^{-2}\bar{Q}_0 G - A^{-1}\bar{Q}_0 H \right\} \\ & + \left\{ -2A^{-3}\bar{Q}_0 G - A^{-2}\bar{Q}_0 H - A^{-1}\bar{Q}_0 I \right\}, \end{aligned} \quad (3.39)$$

We notice again the appearance of secular terms and that no constraints on the solution are imposed for defining the flow, in contrast to the standard Chapman-Enskog expansion method.<sup>18)</sup>

Summing up the perturbative solutions up to the second order, we have an approximate solution around  $\tau \simeq \tau_0$  to this order;

$$\tilde{f}_p(\tau, \sigma; \tau_0) = \tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) + \varepsilon \tilde{f}_p^{(1)}(\tau, \sigma; \tau_0) + \varepsilon^2 \tilde{f}_p^{(2)}(\tau, \sigma; \tau_0) + O(\varepsilon^3). \quad (3.40)$$

We emphasize that this solution contains the secular terms which apparently invalidates the perturbative expansion for  $\tau$  away from the initial time  $\tau_0$ .

### 3.5. RG improvement of perturbative expansion

The point of the RG method lies in the fact that we can utilize the secular terms to obtain an asymptotic solution valid in a global domain. Now we may see that we have a family of curves  $\tilde{f}_p(\tau, \sigma; \tau_0)$  parameterized with  $\tau_0$ . They are all on the exact solution  $f_p(\sigma; \tau)$  at  $\tau = \tau_0$  up to  $O(\varepsilon^3)$ , but only valid locally for  $\tau$  near  $\tau_0$ . So it is conceivable that the envelope of the family of curves which contacts with each local solution at  $\tau = \tau_0$  will give a global solution in our asymptotic situation. According to the classical theory of envelopes, the envelope which contact with any curve in the family at  $\tau = \tau_0$  is obtained by <sup>\*)</sup>

$$\left. \frac{d}{d\tau_0} \tilde{f}_p(\tau, \sigma; \tau_0) \right|_{\tau_0=\tau} = 0, \quad (3.41)$$

or explicitly

$$\begin{aligned} \frac{\partial}{\partial \tau} \left\{ f^{\text{eq}} - \varepsilon A^{-1} \bar{Q}_0 F \right\} - \varepsilon \bar{P}_0 F \\ - \varepsilon^2 \left\{ \bar{P}_0 I - 2A^{-2} \bar{Q}_0 G - A^{-1} \bar{Q}_0 H \right\} + O(\varepsilon^3) = 0. \end{aligned} \quad (3.42)$$

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<sup>\*)</sup> See Appendix B for the foundation of this procedure.

This envelope equation is the basic equation in the RG method and gives the equation of motion governing the dynamics of the five slow variables  $T(\sigma; \tau)$ ,  $\mu(\sigma; \tau)$  and  $u^\mu(\sigma; \tau)$  in  $f_p^{\text{eq}}(\sigma; \tau)$ . The global solution in the asymptotic region is given as an envelope function,

$$\begin{aligned} f_{\text{Ep}}(\tau, \sigma) &\equiv \tilde{f}_p(\tau, \sigma; \tau_0 = \tau) \\ &= f_p^{\text{eq}}(\sigma; \tau) - \varepsilon [A^{-1} \bar{Q}_0 F]_p(\sigma; \tau) \\ &\quad - \varepsilon^2 \left\{ 2 [A^{-3} \bar{Q}_0 G]_p(\sigma; \tau) \right. \\ &\quad \left. + [A^{-2} \bar{Q}_0 H]_p(\sigma; \tau) + [A^{-1} \bar{Q}_0 I]_p(\sigma; \tau) \right\} + O(\varepsilon^3), \end{aligned} \quad (3.43)$$

where the exact solution of Eq.(3.42) is inserted. As is proved in Appendix B, the envelope function  $f_{\text{Ep}}(\tau, \sigma)$  satisfies Eq.(2.28) in a global domain up to  $O(\varepsilon^3)$  owing to the condition (3.41), although  $\tilde{f}_p(\tau, \sigma; \tau_0)$  itself was constructed as a local solution around  $\tau \sim \tau_0$ . Thus, one sees that  $f_{\text{Ep}}(\tau, \sigma)$  now describes a coarse-grained evolution of the one-particle distribution function in Eq.(2.28), because the time-derivatives of the quantities in  $f_{\text{Ep}}(\tau, \sigma)$  are all in the order of  $\varepsilon$  or higher. We emphasize that we have derived the slow-motion equation of Eq.(2.28) in the form of the pair of Eq.'s (3.42) and (3.43).

### 3.6. Reduction of the RG equation to generic hydrodynamic equation

Now let us see that the RG/Envelope equation (3.42) is actually the hydrodynamic equation governing the five slow variables,  $T(\sigma; \tau)$ ,  $\mu(\sigma; \tau)$ , and  $u^\mu(\sigma; \tau)$ . To show this explicitly, we apply  $\bar{P}_0$  from the left and then take the inner product with the five zero modes  $\varphi_0^\alpha$ . In this procedure, we first note that the direct use of the definitions of Eq.'s (3.24), (3.26), and (3.28) leads to the following identity;

$$\sum_p \frac{1}{p^0} (p \cdot \mathbf{a}_p) \varphi_{0p}^\alpha [\bar{P}_0 \psi]_p = \sum_p \frac{1}{p^0} (p \cdot \mathbf{a}_p) \varphi_{0p}^\alpha \psi_p. \quad (3.44)$$

Furthermore, noting that  $\varphi_{0p}^\alpha$  are the collision invariants as shown in Eq.'s (2.7) and (2.8), we have

$$\sum_p \frac{1}{p^0} (p \cdot \mathbf{a}_p) \varphi_{0p}^\alpha \sum_q \sum_r B_{pqr} [A^{-1} \bar{Q}_0 F]_q [A^{-1} \bar{Q}_0 F]_r = 0. \quad (3.45)$$

Here, we have used the relation

$$\sum_q \sum_r B_{pqr} [A^{-1} \bar{Q}_0 F]_q [A^{-1} \bar{Q}_0 F]_r = \frac{1}{p \cdot \mathbf{a}_p} C[A^{-1} \bar{Q}_0 F]_p, \quad (3.46)$$

which follows from the definitions of Eq.'s (2.2) and (3.33).

Thus, we have

$$\sum_p \frac{1}{p^0} \varphi_{0p}^\alpha \left[ (p \cdot \mathbf{a}_p) \frac{\partial}{\partial \tau} + \varepsilon p \cdot \nabla \right] \left\{ f_p^{\text{eq}} - \varepsilon [A^{-1} \bar{Q}_0 F]_p \right\} + O(\varepsilon^3) = 0, \quad (3.47)$$

Putting back  $\varepsilon = 1$ , we arrive at

$$\partial_\mu J_{1st}^{\mu\alpha} = 0, \quad (3.48)$$

with

$$J_{1st}^{\mu\alpha} \equiv \sum_p \frac{1}{p^0} p^\mu \varphi_{0p}^\alpha \left\{ f_p^{\text{eq}} - [A^{-1} \bar{Q}_0 F]_p \right\}, \quad (3.49)$$

where we have used

$$(p \cdot \mathbf{a}_p) \frac{\partial}{\partial \tau} + p \cdot \nabla = p^\mu \partial_\mu. \quad (3.50)$$

It is noted that  $J_{1st}^{\mu\alpha}$  perfectly agrees with the one obtained by inserting the solution  $f_{Ep}(\tau, \sigma)$  in Eq.(3.43) into  $N^\mu$  and  $T^{\mu\nu}$  in Eq.'s (2.10) and (2.11):

$$N^\mu = m^{-1} J_{1st}^{\mu 4}, \quad (3.51)$$

$$T^{\mu\nu} = J_{1st}^{\mu\nu}. \quad (3.52)$$

Therefore, we conclude that Eq.(3.48) is identically the relativistic dissipative hydrodynamic equation: We note that the subscript of “1st” in  $J_{1st}^{\mu\alpha}$  means that the obtained equation is in a class of the so-called *first-order* relativistic dissipative hydrodynamics. As is shown in Appendix A, the first-order equations can be derived phenomenologically with use of the entropy current which includes dissipative effects up to the first order.

It is noted that the hydrodynamic equation (3.48) with the currents (3.49) still contains the macroscopic-frame vector  $\mathbf{a}_p^\mu$ , which is now dependent on  $\tau$  as well as on  $\sigma$ . We shall see that Eq.(3.49) reduces to the currents in various frames with a choice of  $\mathbf{a}_p^\mu$ . In other words, we have obtained the relativistic hydrodynamic equations for a viscous fluid in a generic frame, which is a kind of the master equation from which various relativistic hydrodynamic equations are deduced. As far as we are aware of, this is the first time when such a generic hydrodynamic equation for the relativistic viscous system is obtained. This is one of the main results in the present work. We stress that this was made possible because any ad-hoc ansatz such as the conditions of fit is not necessary in the RG method.

It is also worth mentioning that the problematic Burnett term is absent in Eq.(3.47) thanks to Eq.(3.45). If the Burnett term were to remain, the particle-number and energy-momentum conservation laws are lost; moreover, boundary conditions might have to be taken care of simultaneously because its magnitude is comparable to the Burnett term.<sup>39)</sup> In fact, the presence of the Burnett term is known to be inevitable when the Chapman-Enskog expansion method<sup>39)</sup> is applied to derive the Navier-Stokes equation from the non-relativistic Boltzmann equation <sup>\*)</sup>.

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<sup>\*)</sup> We did not recognize this important point in Ref.36) that the RG method naturally gives a hydrodynamic equation free from the Burnett term. We expect that the mechanism similar to Eq.(3.45) is realized in the non-relativistic case.

We decompose  $J_{1st}^{\mu\alpha}$  into two parts as  $J_{1st}^{\mu\alpha} = J_{1st}^{(0)\mu\alpha} + \delta J_{1st}^{\mu\alpha}$ , where

$$J_{1st}^{(0)\mu\alpha} \equiv \sum_p \frac{1}{p^0} p^\mu \varphi_{0p}^\alpha f_p^{\text{eq}}, \quad (3.53)$$

$$\delta J_{1st}^{\mu\alpha} \equiv - \sum_p \frac{1}{p^0} p^\mu \varphi_{0p}^\alpha [A^{-1} \bar{Q}_0 F]_p = - \langle \tilde{\varphi}_1^{\mu\alpha}, L^{-1} Q_0 f^{\text{eq}-1} F \rangle, \quad (3.54)$$

with

$$\tilde{\varphi}_{1p}^{\mu\alpha} \equiv p^\mu \varphi_{0p}^\alpha \frac{1}{p \cdot \mathbf{a}_p}. \quad (3.55)$$

Needless to say,  $J_{1st}^{(0)\mu\alpha}$  and  $\delta J_{1st}^{\mu\alpha}$  represent the currents in the perfect-fluid and dissipative part, respectively. Corresponding to  $J_{1st}^{(0)\mu\alpha}$  and  $\delta J_{1st}^{\mu\alpha}$ ,  $N^\mu$  and  $T^{\mu\nu}$  in Eq.'s (3.51) and (3.52) are decomposed as

$$N^\mu = N^{(0)\mu} + \delta N^\mu, \quad (3.56)$$

$$T^{\mu\nu} = T^{(0)\mu\nu} + \delta T^{\mu\nu}, \quad (3.57)$$

where

$$N^{(0)\mu} \equiv m^{-1} J_{1st}^{(0)\mu 4}, \quad (3.58)$$

$$\delta N^\mu \equiv m^{-1} \delta J_{1st}^{\mu 4}, \quad (3.59)$$

$$T^{(0)\mu\nu} \equiv J_{1st}^{(0)\mu\nu}, \quad (3.60)$$

$$\delta T^{\mu\nu} \equiv \delta J_{1st}^{\mu\nu}. \quad (3.61)$$

For later convenience, we present a simpler form of  $\delta J_{1st}^{\mu\alpha}$ . First, we note that  $F$  in Eq.(3.13) is expressed as

$$F_p = -f_p^{\text{eq}} \left( \tilde{\varphi}_{1p}^{\mu 4} m^{-1} \nabla_\mu \frac{\mu}{T} - \tilde{\varphi}_{1p}^{\mu\nu} \nabla_\mu \frac{u_\nu}{T} \right) = -f_p^{\text{eq}} \tilde{\varphi}_{1p}^{\mu\alpha} \bar{X}_{\mu\alpha}, \quad (3.62)$$

where

$$\bar{X}_{\mu\alpha} \equiv \begin{cases} -\nabla_\mu (u_\nu/T) & \text{for } \alpha = \nu, \\ m^{-1} \nabla_\mu (\mu/T) & \text{for } \alpha = 4. \end{cases} \quad (3.63)$$

Using the above representation of  $F$  and the identity

$$\langle \varphi, L^{-1} Q_0 \psi \rangle = \langle Q_0 \varphi, L^{-1} Q_0 \psi \rangle, \quad (3.64)$$

we can reduce the dissipative part  $\delta J_{1st}^{\mu\alpha}$  to the following form:

$$\delta J_{1st}^{\mu\alpha} = \eta_1^{\mu\alpha\nu\beta} \bar{X}_{\nu\beta}, \quad (3.65)$$

where

$$\eta_1^{\mu\alpha\nu\beta} \equiv \langle \varphi_1^{\mu\alpha}, L^{-1} \varphi_1^{\nu\beta} \rangle. \quad (3.66)$$

Here, we have introduced an important new vector defined by

$$\varphi_{1p}^{\mu\alpha} \equiv [Q_0 \tilde{\varphi}_1^{\mu\alpha}]_p. \quad (3.67)$$

We call  $\varphi_{1p}^{\mu\alpha}$  the *first-excited modes*. It is noteworthy that  $\delta J_{1st}^{\mu\alpha}$  is represented as a product of  $\eta_1^{\mu\alpha\nu\beta}$  and  $\bar{X}_{\nu\beta}$ :  $\eta_1^{\mu\alpha\nu\beta}$  has some information about the transport coefficients, while  $\bar{X}_{\nu\beta}$  is identical to the corresponding thermodynamic forces.

Finally, we write down the relativistic dissipative hydrodynamic equation defined in the covariant coordinate system  $(\sigma; \tau)$ :

$$\sum_p \frac{1}{p^0} \varphi_{0p}^\alpha \left[ (p \cdot \mathbf{a}_p) \frac{\partial}{\partial \tau} + p \cdot \nabla \right] \left[ f_p^{\text{eq}} \left( 1 + [L^{-1} \varphi_1^{\nu\beta}]_p \bar{X}_{\nu\beta} \right) \right] = 0, \quad (3.68)$$

which can be derived straightforwardly from Eq.'s (3.47) and (3.62). This form of the relativistic dissipative hydrodynamic equation will be found to play an essential role in the stability analysis to be presented in §5.3.

#### §4. Relativistic Hydrodynamic Equation with Specific Macroscopic-frame Vectors

In this section, we give explicit forms of the relativistic dissipative hydrodynamic equations by integrating out the right-hand side of Eq.'s (3.53) and (3.54) with respect to the momentum  $p^\mu$ .

A remark is in order here: We calculate and present the thermodynamic quantities and transport coefficients using our model equation, i.e., the relativistic Boltzmann equation, for completeness. Then, the explicit forms of them are inherently for the relativistic rarefied gas. We would like to remind the reader, however, that the main purpose of the present work is to determine the form of the relativistic hydrodynamic equations for a viscous fluid, and expect that the forms of the macroscopic hydrodynamic equations which contain the thermodynamic quantities and transport coefficients only parametrically, and hence the forms are independent of the microscopic expressions of these quantities.

After the integration, the currents of perfect-fluid part  $J_{1st}^{(0)\mu\alpha}$  reads

$$J_{1st}^{(0)\mu\alpha} = \begin{cases} e u^\mu u^\nu - p \Delta^{\mu\nu} = T^{(0)\mu\nu} & \text{for } \alpha = \nu, \\ m n u^\mu = m N^{(0)\mu} & \text{for } \alpha = 4, \end{cases} \quad (4.1)$$

where  $e$ ,  $p$ , and  $n$  denote the internal energy, the pressure, and the particle-number density, respectively. These quantities are defined by

$$n \equiv \sum_p \frac{1}{p^0} f_p^{\text{eq}} (p \cdot u) = (2\pi)^{-3} 4\pi m^3 e^{\frac{\mu}{T}} z^{-1} K_2(z), \quad (4.2)$$

$$e \equiv \sum_p \frac{1}{p^0} f_p^{\text{eq}} (p \cdot u)^2 = m n \left[ \frac{K_3(z)}{K_2(z)} - z^{-1} \right], \quad (4.3)$$

$$p \equiv \sum_p \frac{1}{p^0} f_p^{\text{eq}} (-1/3 p^\mu p^\nu \Delta_{\mu\nu}) = n T, \quad (4.4)$$

where we have introduced the second- and third-order modified Bessel functions  $K_2(z)$  and  $K_3(z)$  with  $z$  being the dimensionless variable defined by

$$z \equiv \frac{m}{T}. \quad (4.5)$$

The explicit form of the modified Bessel functions is presented in Eq.(C.8) in Appendix C.

#### 4.1. A generic choice of the macroscopic-frame vector

The dissipative part of the currents  $\delta J_{1st}^{\mu\alpha}$  depends on the macroscopic-frame vector  $\mathbf{a}_p^\mu$  explicitly, in the contrast of  $J_{1st}^{(0)\mu\alpha}$ . Here, we shall consider  $\delta J_{1st}^{\mu\alpha}$  with a choice of  $\mathbf{a}_p^\mu$ . As a simple but nontrivial choice, let us take the following set of the macroscopic-frame vectors with  $\theta$  being a constant;

$$\mathbf{a}_p^\mu = \frac{1}{p \cdot u} \left( (p \cdot u) \cos \theta + m \sin \theta \right) u^\mu \equiv \theta_p^\mu. \quad (4.6)$$

In Eq.(4.6), we note that the factor  $m$  is introduced simply to make the expression dimensionless, so our method is also applicable to the case of massless particles.

A remark is in order here: The inequality (2.18) of  $\mathbf{a}_p^0$  leads to a restriction on  $\theta$  when  $\mathbf{a}_p^\mu = \theta_p^\mu$  in Eq.(3.19), as  $(p \cdot u) \cos \theta + m \sin \theta = \sqrt{(p \cdot u)^2 + m^2} \sin(\theta + \chi) > 0$  with  $\tan \chi = (p \cdot u)/m \geq 1$ , which implies that

$$-\frac{\pi}{4} < \theta \leq \frac{\pi}{2}, \quad (4.7)$$

because  $\pi/4 \leq \chi < \pi/2$ .

Under these settings, we shall now give the explicit representation of  $\delta J_{1st}^{\mu\alpha} = \eta_1^{\mu\alpha\nu\beta} \bar{X}_{\nu\beta}$ . First, we consider  $\bar{X}_{\mu\alpha}$  in Eq.(3.65). Inserting the equality  $\mathbf{a}_p^\mu = \theta_p^\mu$ , we have

$$\Delta_p^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu = \Delta^{\mu\nu}, \quad (4.8)$$

so that the differential operator with respect to  $\sigma^\mu$  reads

$$\nabla^\mu = \Delta^{\mu\nu} \partial_\nu \equiv \nabla^\mu. \quad (4.9)$$

Notice that  $\Delta^{\mu\nu}$  and  $\nabla^\mu$  are now identical to the familiar projection matrix onto the subspace complement to  $u^\mu$  and the covariant spatial differential operator,<sup>18)</sup> respectively. Thus, the thermodynamic forces  $\bar{X}_{\mu\alpha}$  are reduced to

$$\bar{X}_{\mu\alpha} = \begin{cases} -\nabla_\mu(u_\nu/T) & \text{for } \alpha = \nu, \\ m^{-1} \nabla_\mu(\mu/T) & \text{for } \alpha = 4. \end{cases} \quad (4.10)$$

Next, we derive an explicit form of  $\eta_1^{\mu\alpha\nu\beta}$  in Eq.(3.65), which task is tantamount to obtaining an explicit representation of  $\varphi_{1p}^{\mu\alpha}$ . After a straightforward calculation

presented in Appendix C, one can find that  $\varphi_{1p}^{\mu\alpha}$  are written as

$$\varphi_{1p}^{\mu\alpha} = \begin{cases} \tilde{\Pi}_p \left( Y_1(\theta) u^\mu u^\nu - Y_2(\theta) \Delta^{\mu\nu} \right) \\ \quad + \tilde{J}_p^\mu Y_3(\theta) u^\nu + \tilde{J}_p^\nu Y_3(\theta) u^\mu + \tilde{\pi}_p^{\mu\nu} & \text{for } \alpha = \nu, \\ \tilde{\Pi}_p Z_1(\theta) u^\mu + \tilde{J}_p^\mu Z_2(\theta) & \text{for } \alpha = 4. \end{cases} \quad (4.11)$$

Here, we have introduced  $\tilde{\Pi}_p$ ,  $\tilde{J}_p^\mu$ , and  $\tilde{\pi}_p^{\mu\nu}$  defined by

$$(\tilde{\Pi}_p, \tilde{J}_p^\mu, \tilde{\pi}_p^{\mu\nu}) \equiv (\Pi_p, J_p^\mu, \pi_p^{\mu\nu}) \frac{1}{p \cdot \theta_p}, \quad (4.12)$$

$$\Pi_p \equiv \left( \frac{4}{3} - \gamma \right) (p \cdot u)^2 + \left( (\gamma - 1) T \hat{h} - \gamma T \right) (p \cdot u) - \frac{1}{3} m^2, \quad (4.13)$$

$$J_p^\mu \equiv - \left( (p \cdot u) - T \hat{h} \right) \Delta^{\mu\nu} p_\nu, \quad (4.14)$$

$$\pi_p^{\mu\nu} \equiv \Delta^{\mu\nu\rho\sigma} p_\rho p_\sigma, \quad (4.15)$$

and  $Y_1(\theta)$ ,  $Y_2(\theta)$ ,  $Y_3(\theta)$ ,  $Z_1(\theta)$ , and  $Z_2(\theta)$  defined by

$$Y_1(\theta) \equiv \frac{-3 z^2 \sin^2 \theta}{z^2 \cos^2 \theta + z^2 (3\gamma - 4) \sin^2 \theta - 3z [1 - (\hat{h} - 1)(\gamma - 1)] \cos \theta \sin \theta}, \quad (4.16)$$

$$Y_2(\theta) \equiv \frac{z^2 \cos^2 \theta - z^2 \sin^2 \theta}{z^2 \cos^2 \theta + z^2 (3\gamma - 4) \sin^2 \theta - 3z [1 - (\hat{h} - 1)(\gamma - 1)] \cos \theta \sin \theta}, \quad (4.17)$$

$$Y_3(\theta) \equiv \frac{-z \sin \theta}{\hat{h} \cos \theta + z \sin \theta}, \quad (4.18)$$

$$Z_1(\theta) \equiv \frac{3 z^2 \sin \theta \cos \theta}{z^2 \cos^2 \theta + z^2 (3\gamma - 4) \sin^2 \theta - 3z [1 - (\hat{h} - 1)(\gamma - 1)] \cos \theta \sin \theta}, \quad (4.19)$$

$$Z_2(\theta) \equiv \frac{z \cos \theta}{\hat{h} \cos \theta + z \sin \theta}, \quad (4.20)$$

respectively. It is noted that  $\tilde{\Pi}_p$ ,  $\tilde{J}_p^\mu$ , and  $\tilde{\pi}_p^{\mu\nu}$  belong to the  $\mathbb{Q}_0$  space.  $\Pi_p$ ,  $J_p^\mu$ , and  $\pi_p^{\mu\nu}$  are independent of  $\theta$ , which are identically the *microscopic representations of dissipative currents* in the literature.<sup>18)</sup> The definitions of  $\hat{h}$ ,  $\gamma$ , and  $\Delta^{\mu\nu\rho\sigma}$  used in Eq.'s (4.13)-(4.20) are

$$\hat{h} \equiv \frac{e + p}{n T}, \quad (4.21)$$

$$\gamma \equiv 1 + (z^2 - \hat{h}^2 + 5\hat{h} - 1)^{-1}, \quad (4.22)$$

$$\Delta^{\mu\nu\rho\sigma} \equiv \frac{1}{2} \left( \Delta^{\mu\rho} \Delta^{\nu\sigma} + \Delta^{\mu\sigma} \Delta^{\nu\rho} - \frac{2}{3} \Delta^{\mu\nu} \Delta^{\rho\sigma} \right), \quad (4.23)$$



where  $\hat{h}$  and  $\gamma$  denote the reduced enthalpy per particle and the ratio of the heat capacities, respectively.

With use of  $\varphi_{1p}^{\mu\alpha}$  in Eq.(4.11), we can write down  $\eta_1^{\mu\alpha\nu\beta} = \langle \varphi_1^{\mu\alpha}, L^{-1} \varphi_1^{\nu\beta} \rangle$  as

$$\begin{aligned} \eta_1^{\mu\rho\nu\sigma} = & -T \zeta \left( Y_1(\theta) u^\mu u^\rho - Y_2(\theta) \Delta^{\mu\rho} \right) \left( Y_1(\theta) u^\nu u^\sigma - Y_2(\theta) \Delta^{\nu\sigma} \right) \\ & + T^2 \lambda Y_3^2(\theta) (u^\mu u^\nu \Delta^{\rho\sigma} + u^\mu u^\sigma \Delta^{\rho\nu} + u^\rho u^\nu \Delta^{\mu\sigma} + u^\rho u^\sigma \Delta^{\mu\nu}) \\ & - 2T \eta \Delta^{\mu\rho\nu\sigma}, \end{aligned} \quad (4.24)$$

$$\begin{aligned} \eta_1^{\mu\rho\nu 4} = \eta_1^{\nu 4\mu\rho} = & -T \zeta \left( Y_1(\theta) u^\mu u^\rho - Y_2(\theta) \Delta^{\mu\rho} \right) Z_1(\theta) u^\nu \\ & + T^2 \lambda Y_3(\theta) Z_2(\theta) (u^\mu \Delta^{\rho\nu} + u^\rho \Delta^{\mu\nu}), \end{aligned} \quad (4.25)$$

$$\eta_1^{\mu 4\nu 4} = -T \zeta Z_1^2(\theta) u^\mu u^\nu + T^2 \lambda Z_2^2(\theta) \Delta^{\mu\nu}. \quad (4.26)$$

Here, we have introduced the transport coefficients, i.e., the bulk viscosity  $\zeta$ , the heat conductivity  $\lambda$ , and the shear viscosity  $\eta$  given by

$$\zeta \equiv -\frac{1}{T} \langle \tilde{\Pi}, L^{-1} \tilde{\Pi} \rangle, \quad (4.27)$$

$$\lambda \equiv \frac{1}{3T^2} \langle \tilde{J}^\mu, L^{-1} \tilde{J}_\mu \rangle, \quad (4.28)$$

$$\eta \equiv -\frac{1}{10T} \langle \tilde{\pi}^{\mu\nu}, L^{-1} \tilde{\pi}_{\mu\nu} \rangle, \quad (4.29)$$

respectively. In §5.1, we show that these quantities are independent of  $\theta$ . In the derivations of Eq.'s (4.24)-(4.26), we have used the following identities

$$\langle \tilde{J}^\mu, L^{-1} \tilde{J}^\nu \rangle = \frac{1}{3} \langle \tilde{J}^a, L^{-1} \tilde{J}_a \rangle \Delta^{\mu\nu}, \quad (4.30)$$

$$\langle \tilde{\pi}^{\mu\nu}, L^{-1} \tilde{\pi}^{\rho\sigma} \rangle = \frac{1}{5} \langle \tilde{\pi}^{ab}, L^{-1} \tilde{\pi}_{ab} \rangle \Delta^{\mu\nu\rho\sigma}. \quad (4.31)$$

Now let us rewrite the expressions of  $\zeta$ ,  $\lambda$ , and  $\eta$  in a more familiar form, i.e., the Green-Kubo formula<sup>40)</sup> in the linear response theory.<sup>24), 41)</sup> With use of the identity

$$\sum_q [L^{-1}]_{pq} (\tilde{\Pi}_q, \tilde{J}_q^\mu, \tilde{\pi}_q^{\mu\nu}) = -\int_0^\infty ds \sum_q [e^{sL}]_{pq} (\tilde{\Pi}_q, \tilde{J}_q^\mu, \tilde{\pi}_q^{\mu\nu}), \quad (4.32)$$

we can rewrite Eq's (4.27)-(4.29) as

$$\zeta = \int_0^\infty ds R_\zeta(s), \quad (4.33)$$

$$\lambda = \int_0^\infty ds R_\lambda(s), \quad (4.34)$$

$$\eta = \int_0^\infty ds R_\eta(s), \quad (4.35)$$

where

$$R_\zeta(s) \equiv \frac{1}{T} \langle \tilde{\Pi}(0), \tilde{\Pi}(s) \rangle, \quad (4.36)$$

$$R_\lambda(s) \equiv -\frac{1}{3T^2} \langle \tilde{J}^\mu(0), \tilde{J}_\mu(s) \rangle, \quad (4.37)$$

$$R_\eta(s) \equiv \frac{1}{10T} \langle \tilde{\pi}^{\mu\nu}(0), \tilde{\pi}_{\mu\nu}(s) \rangle, \quad (4.38)$$

with

$$(\tilde{H}_p(s), \tilde{J}_p^\mu(s), \tilde{\pi}_p^{\mu\nu}(s)) \equiv \sum_q [e^{sL}]_{pq} (\tilde{H}_q, \tilde{J}_q^\mu, \tilde{\pi}_q^{\mu\nu}). \quad (4.39)$$

It is noted that  $R_\zeta(s)$ ,  $R_\lambda(s)$ , and  $R_\eta(s)$  in Eq.'s (4.36)-(4.38) are called the *relaxation function* in the linear response theory.<sup>24),41)</sup>

Finally, the dissipative currents  $\delta J_{1st}^{\mu\alpha}$  are obtained from  $\eta_1^{\mu\alpha\nu\beta}$  in Eq.'s (4.24)-(4.26) and  $\bar{X}_{\nu\beta}$  in Eq.(4.10), as follows:

$$\delta J_{1st}^{\mu\alpha} = \begin{cases} \zeta \left( Y_1(\theta) u^\mu u^\nu - Y_2(\theta) \Delta^{\mu\nu} \right) X_\Pi \\ \quad - T \lambda Y_3(\theta) (u^\mu X_J^\nu + u^\nu X_J^\mu) + 2\eta X_\pi^{\mu\nu} = \delta T^{\mu\nu} & \text{for } \alpha = \nu, \\ \zeta Z_1(\theta) u^\mu X_\Pi - T \lambda Z_2(\theta) X_J^\mu = m \delta N^\mu & \text{for } \alpha = 4, \end{cases} \quad (4.40)$$

where the new thermodynamic forces  $X_\Pi$ ,  $X_J^\mu$ , and  $X_\pi^{\mu\nu}$  are defined by

$$\begin{aligned} X_\Pi &\equiv -T \left( Y_1(\theta) u^\mu u^\nu - Y_2(\theta) \Delta^{\mu\nu} \right) \bar{X}_{\mu\nu} - T Z_1(\theta) u^\mu \bar{X}_{\mu 4} \\ &= -Y_2(\theta) \nabla \cdot u, \end{aligned} \quad (4.41)$$

$$\begin{aligned} X_J^\mu &\equiv -T \Delta^{\mu\nu} \left( Y_3(\theta) u^\rho (\bar{X}_{\nu\rho} + \bar{X}_{\rho\nu}) + Z_2(\theta) \bar{X}_{\nu 4} \right) \\ &= Y_3(\theta) T \nabla_\mu (1/T) - Z_2(\theta) z^{-1} \nabla_\mu (\mu/T), \end{aligned} \quad (4.42)$$

$$\begin{aligned} X_\pi^{\mu\nu} &\equiv -T \Delta^{\mu\nu\rho\sigma} \bar{X}_{\rho\sigma} \\ &= \Delta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma. \end{aligned} \quad (4.43)$$

Here, the following relations have been used:  $u^\mu \bar{X}_{\mu\alpha} = 0$  and  $u^\nu \nabla_\mu u_\nu = 0$ . Thus, we have arrived at the generic form of the currents  $J_{1st}^{\mu\alpha} = J_{1st}^{(0)\mu\alpha} + \delta J_{1st}^{\mu\alpha}$  for an arbitrary  $\theta$  specifying the local rest frame of the flow. Equations (4.1) and (4.40) together with Eq.'s (4.41)-(4.43) are the main results in this section.

To have a physical intuition into the effects of the dissipation on the hydrodynamic equations, we also decompose  $\delta J_{1st}^{\mu\alpha}$  in Eq.(4.40) into various tensors

$$\delta J_{1st}^{\mu\alpha} = \begin{cases} \delta e u^\mu u^\nu - \delta p \Delta^{\mu\nu} + Q^\mu u^\nu + Q^\nu u^\mu + \Pi^{\mu\nu} = \delta T^{\mu\nu} & \text{for } \alpha = \nu, \\ m \delta n u^\mu + m \nu^\mu = m \delta N^\mu, & \text{for } \alpha = 4, \end{cases} \quad (4.44)$$

where

$$\delta e \equiv \delta J_{1st}^{\mu\nu} u_\mu u_\nu = \delta T^{\mu\nu} u_\mu u_\nu, \quad (4.45)$$

$$\delta p \equiv -1/3 \delta J_{1st}^{\mu\nu} \Delta_{\mu\nu} = -1/3 \delta T^{\mu\nu} \Delta_{\mu\nu}, \quad (4.46)$$

$$Q^\mu \equiv \delta J_{1st}^{\nu\rho} u_\nu \Delta_\rho^\mu = \delta T^{\nu\rho} u_\nu \Delta_\rho^\mu, \quad (4.47)$$

$$\Pi^{\mu\nu} \equiv \delta J_{1st}^{\rho\sigma} \Delta_{\rho\sigma}^{\mu\nu} = \delta T^{\rho\sigma} \Delta_{\rho\sigma}^{\mu\nu}, \quad (4.48)$$

$$\delta n \equiv m^{-1} \delta J_{1st}^{\mu 4} u_\mu = \delta N^\mu u_\mu, \quad (4.49)$$

$$\nu^\mu \equiv m^{-1} \delta J_{1st}^{\nu 4} \Delta_\nu^\mu = \delta N^\nu \Delta_\nu^\mu. \quad (4.50)$$

Here,  $\delta e$ ,  $\delta p$ , and  $\delta n$  represent the contribution of the dissipation to the internal energy, pressure and particle-number density, respectively. We stress that the dissipation in the relativistic system generically gives rise to an additional contribution to such would-be thermodynamic quantities, as well as the dissipative currents,  $Q^\mu$ ,  $\Pi^{\mu\nu}$ , and  $\nu^\mu$ . Furthermore, it is to be noted that the existence of  $\delta e$  defined in Eq.(4.45) is directly related with the definition of the local rest frame as is seen from Eq.(1.1).

Inserting Eq.(4.40) into Eq.'s (4.45)-(4.50), we have

$$\begin{aligned} \delta e &= \zeta Y_1(\theta) X_\Pi \\ &= -\zeta Y_1(\theta) Y_2(\theta) \nabla \cdot u, \end{aligned} \quad (4.51)$$

$$\begin{aligned} \delta p &= \zeta Y^2(\theta) X_\Pi \\ &= -\zeta Y_2^2(\theta) \nabla \cdot u, \end{aligned} \quad (4.52)$$

$$\begin{aligned} Q^\mu &= -T \lambda Y_3(\theta) X_J^\mu \\ &= -T \lambda \left( Y_3^2(\theta) T \nabla^\mu (1/T) - Y_3(\theta) Z_2(\theta) z^{-1} \nabla^\mu (\mu/T) \right), \end{aligned} \quad (4.53)$$

$$\begin{aligned} \Pi^{\mu\nu} &= 2\eta X_\pi^{\mu\nu} \\ &= 2\eta \Delta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma, \end{aligned} \quad (4.54)$$

$$\begin{aligned} m \delta n &= \zeta Z_1(\theta) X_\Pi \\ &= -\zeta Y_2(\theta) Z_1(\theta) \nabla \cdot u, \end{aligned} \quad (4.55)$$

$$\begin{aligned} m \nu^\mu &= -T \lambda Z_2(\theta) X_J^\mu \\ &= -T \lambda \left( Y_3(\theta) Z_2(\theta) T \nabla^\mu (1/T) - Z_2^2(\theta) z^{-1} \nabla^\mu (\mu/T) \right). \end{aligned} \quad (4.56)$$

Equations (4.51)-(4.56) also constitute main results in this section, and we call them the *constitutive equations*.

We shall consider here the constitutive equations only for a few values of  $\theta$  which satisfies Eq.(4.7), but show that our generic form of the constitutive equations contains ones in the two well-known frames, i.e., the energy one and the particle one, which were introduced in §1.

#### 4.2. Energy frame

With the simplest choice  $\theta = 0$ , i.e.,  $\mathbf{a}_p^\mu = u^\mu$ , we have

$$\delta e = 0, \quad (4.57)$$

$$\delta p = -\zeta \nabla \cdot u, \quad (4.58)$$

$$Q^\mu = 0, \quad (4.59)$$

$$\Pi^{\mu\nu} = 2\eta \Delta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma, \quad (4.60)$$

$$m \delta n = 0, \quad (4.61)$$

$$m \nu^\mu = \lambda \frac{m}{\hat{h}^2} \nabla^\mu \frac{\mu}{T}, \quad (4.62)$$

where we have used the following relations obtained from Eq.'s (4.16)-(4.20),

$$Y_1(0) = 0, \quad (4.63)$$

$$Y_2(0) = 1, \quad (4.64)$$

$$Y_3(0) = 0, \quad (4.65)$$

$$Z_1(0) = 0, \quad (4.66)$$

$$Z_2(0) = z/\hat{h}. \quad (4.67)$$

It is noted that the choice  $\theta = 0$  gives  $Q^\mu = 0$  as well as  $\nu^\mu \neq 0$ , which tells us that our hydrodynamic equation with  $\theta = 0$  is the one in the energy frame. We remark that the dissipation gives rise to an additional pressure  $\delta p$  while the dissipative internal energy  $\delta e$  and particle-number density  $\delta n$  are absent in this frame. Thus, the conditions of fit (2.38) used in the usual Chapman-Enskog expansion method is found to be compatible with the underlying Boltzmann equation.

Using Eq.'s (4.1) and (4.44), we can rewrite  $J_{1st}^{\mu\alpha} = J_{1st}^{(0)\mu\alpha} + \delta J_{1st}^{\mu\alpha}$  as

$$T^{\mu\nu} = e u^\mu u^\nu - (p - \zeta \nabla \cdot u) \Delta^{\mu\nu} + 2\eta \Delta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma, \quad (4.68)$$

$$N^\mu = n u^\mu + \lambda \frac{1}{\hat{h}^2} \nabla^\mu \frac{\mu}{T}. \quad (4.69)$$

We should notice that  $T^{\mu\nu}$  and  $N^\mu$  in Eq.'s (4.68) and (4.69) completely agree with the energy-frame currents proposed by Landau and Lifshitz.<sup>12)</sup> Indeed, the respective dissipative parts  $\delta T^{\mu\nu}$  and  $\delta N^\mu$  in Eq.'s (4.68) and (4.69) meet Landau-Lifshitz's constraints given by Eq.'s (1.1), (1.2), and (1.4). This was actually anticipated: In fact, if we take the simplest choice  $\mathbf{a}_p^\mu = u^\mu$ , Eq.(2.31) tells us that the physical quantity transported by each particle in the system reads

$$p \cdot u. \quad (4.70)$$

To make clearer the physical meaning of  $(p \cdot u)$ , we take the non-relativistic limit of this quantity:

$$p \cdot u \sim m + \frac{m}{2} \left| \frac{\mathbf{p}}{m} - \mathbf{u} \right|^2, \quad (4.71)$$

where  $u^\mu = (u^0, \mathbf{u})$  and  $p^\mu = (p^0, \mathbf{p})$ . Equation (4.71) shows that  $(p \cdot u)$  can be identified as the kinetic energy of the fluid component measured in the rest frame of  $u^\mu$ . Thus, it is natural that the currents  $J_{1st}^{\mu\alpha}$  with the choice  $\mathbf{a}_p^\mu = u^\mu$  becomes ones in the energy frame adopted by Landau and Lifshitz.

#### 4.3. Particle frame

Another simple choice  $\theta = \pi/2$ , i.e.,  $\mathbf{a}_p^\mu = m/(p \cdot u) u^\mu$ , gives the following constitutive equations

$$\delta e = -3\zeta (3\gamma - 4)^{-2} \nabla \cdot u, \quad (4.72)$$

$$\delta p = -\zeta (3\gamma - 4)^{-2} \nabla \cdot u, \quad (4.73)$$

$$Q^\mu = \lambda \nabla^\mu T, \quad (4.74)$$

$$\Pi^{\mu\nu} = 2\eta \Delta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma, \quad (4.75)$$

$$m \delta n = 0, \quad (4.76)$$

$$m \nu^\mu = 0. \quad (4.77)$$

We have utilized the following relations

$$Y_1(\pi/2) = -3(3\gamma - 4)^{-1}, \quad (4.78)$$

$$Y_2(\pi/2) = -(3\gamma - 4)^{-1}, \quad (4.79)$$

$$Y_3(\pi/2) = 1, \quad (4.80)$$

$$Z_1(\pi/2) = 0, \quad (4.81)$$

$$Z_2(\pi/2) = 0, \quad (4.82)$$

which are derived from Eq.'s (4.16)-(4.20), respectively. We see that  $\nu^\mu = 0$  while  $Q^\mu \neq 0$ , which tells us that our generic equation with  $\theta = \pi/2$  becomes the hydrodynamic equation in the particle frame or Eckart frame. We also note that the dissipative internal energy  $\delta e$  and pressure  $\delta p$  are finite in this frame in our formalism while  $\delta n = 0$ ; we remark that the presence of  $\delta p$  was also the case in the energy frame.

In terms of  $T^{\mu\nu}$  and  $N^\mu$ , we have

$$\begin{aligned} T^{\mu\nu} = & \left( e - 3\zeta (3\gamma - 4)^{-2} \nabla \cdot u \right) u^\mu u^\nu - \left( p - \zeta (3\gamma - 4)^{-2} \nabla \cdot u \right) \Delta^{\mu\nu} \\ & + \lambda (u^\mu \nabla^\nu T + u^\nu \nabla^\mu T) + 2\eta \Delta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma, \end{aligned} \quad (4.83)$$

$$N^\mu = n u^\mu. \quad (4.84)$$

It is noteworthy that the set of Eq.'s (4.83) and (4.84) is different from the phenomenological equations in the particle frame by Eckart, but may be regarded as a corrected version of that by Stewart,<sup>14)</sup> which is derived from the relativistic Boltzmann equation on the basis of the moment method: Indeed, the respective dissipative parts  $\delta T^{\mu\nu}$  and  $\delta N^\mu$  in Eq.'s (4.83) and (4.84) satisfy the conditions (1.5), (1.2), and (1.3) as Stewart's equation does, but not Eckart's constraints given by Eq.'s (1.1), (1.2), and (1.3). The latter also implies that the conditions of fit (2.39) imposed in the usual Chapman-Enskog expansion method is not compatible with the underlying Boltzmann equation.

Now it is interesting that the above constraints on the energy-momentum tensor do not necessarily determine the detailed structure of it. In fact, Eq.(4.83) is different from that of Stewart, which is obtained from our equation through the replacements

$$-\zeta (3\gamma - 4)^{-2} \nabla \cdot u \rightarrow +\zeta (3\gamma - 4)^{-1} \nabla \cdot u, \quad (4.85)$$

$$\lambda \nabla^\mu T \rightarrow \lambda (\nabla^\mu T - T D u^\mu), \quad (4.86)$$

with  $D \equiv u^\mu \partial_\mu$ , although both of them satisfy the constraints (1.5), (1.2), and (1.3). There are two noteworthy points in Eq.(4.83): First, the sign of the thermodynamic

force with the bulk viscosity in our equation is the same as that in the Landau-Lifshitz equation and opposite that in the Stewart equation, because  $\gamma > 4/3$ .<sup>18)</sup> Second, the heat flow  $\Delta_{\mu\nu} u_\rho \delta T^{\nu\rho} = Q^\mu$  does not contain the time-derivative term,  $Du_\mu$ , i.e., the acceleration term, which is often interpreted as one of the important relativistic effects.<sup>11)</sup>

Now a natural question is which equation is preferable, ours or Stewart's or Eckart's? It is known that there have been no hydrodynamic equation including Eckart's and Stewart's which has the stable thermal equilibrium state, as is shown by Lindblum and Hiskock.<sup>17)</sup> Owing to the absence of  $Du^\mu$ , we can show that the thermal equilibrium solution of our equation becomes stable against a small perturbation,<sup>37)</sup> in contrast to the Eckart and the original Stewart equations. Thus, we claim that our equation (4.83) is the proper hydrodynamic equation free from the instability problem in the particle frame. We emphasize that such an equation has been obtained for the first time, as far as we are aware of. A detailed and general account of this stability issue will be given in §5.3.

#### 4.4. A new frame

The other interesting choice is  $\theta = \pi/4$ , for which we have  $\mathbf{a}_p^\mu = ((p \cdot u) + m)/(\sqrt{2} p \cdot u) u^\mu$ , and

$$\delta e = 0, \quad (4.87)$$

$$\delta p = 0, \quad (4.88)$$

$$Q^\mu = \lambda \left( \frac{z}{\hat{h} + z} \right)^2 \left( \nabla^\mu T - z^{-1} T \nabla^\mu \frac{\mu}{T} \right), \quad (4.89)$$

$$\Pi^{\mu\nu} = 2\eta \Delta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma, \quad (4.90)$$

$$m \delta n = 0, \quad (4.91)$$

$$m \nu^\mu = -\lambda \left( \frac{z}{\hat{h} + z} \right)^2 \left( \nabla^\mu T - z^{-1} T \nabla^\mu \frac{\mu}{T} \right). \quad (4.92)$$

Here, we have used the following relations obtained from Eq.'s (4.16)-(4.20),

$$Y_1(\pi/4) = -\frac{3z^2}{z^2 + z^2(3\gamma - 4) - 3z[1 - (\hat{h} - 1)(\gamma - 1)]}, \quad (4.93)$$

$$Y_2(\pi/4) = 0, \quad (4.94)$$

$$Y_3(\pi/4) = -\frac{z}{\hat{h} + z}, \quad (4.95)$$

$$Z_1(\pi/4) = \frac{3z^2}{z^2 + z^2(3\gamma - 4) - 3z[1 - (\hat{h} - 1)(\gamma - 1)]}, \quad (4.96)$$

$$Z_2(\pi/4) = \frac{z}{\hat{h} + z}. \quad (4.97)$$

We note that the present choice  $\theta = \pi/4$  gives a local rest frame where  $Q^\mu \neq 0$  and  $\nu^\mu \neq 0$ ; recall that  $Q^\mu = 0$  ( $\nu^\mu = 0$ ) in the energy (particle) frame. Thus, our generic hydrodynamic equation with  $\theta = \pi/4$  gives the equation in a local rest frame which is neither the energy frame nor the particle frame. We also remark that the

dissipative internal energy, pressure, and particle-number density are absent in this frame;  $\delta e = \delta p = \delta n = 0$ .

The alternative expressions in terms of  $T^{\mu\nu}$  and  $N^\mu$  read

$$\begin{aligned} T^{\mu\nu} = & e u^\mu u^\nu - p \Delta^{\mu\nu} \\ & + \lambda \left( \frac{z}{\hat{h} + z} \right)^2 \left[ u^\mu \left( \nabla^\nu T - z^{-1} T \nabla^\nu \frac{\mu}{T} \right) + u^\nu \left( \nabla^\mu T - z^{-1} T \nabla^\mu \frac{\mu}{T} \right) \right] \\ & + 2 \eta \Delta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma, \end{aligned} \quad (4.98)$$

$$N^\mu = n u^\mu - \lambda \left( \frac{z}{\hat{h} + z} \right)^2 \left( \nabla^\mu T - z^{-1} T \nabla^\mu \frac{\mu}{T} \right). \quad (4.99)$$

We see another unique feature of this frame that the bulk pressure term is absent, and hence  $\zeta$  does not play any role in this frame. As far as we know, this type of the relativistic dissipative hydrodynamics is written down for the first time, which was made possible by the introduction of the macroscopic-frame vector in the powerful RG method <sup>\*)</sup>.

## §5. Discussions

In this section, we shall examine the properties of the derived equation in the generic frame and its advantageous nature over the existing equations in the literature, focusing on transport coefficients and local rest frames. We give a general proof without recourse to numerical calculations that the hydrodynamic equations obtained in our formalism have the stable equilibrium state on the basis of the positive definiteness of the inner product.

### 5.1. Frame independence of transport coefficients

In this subsection, we shall show that the transport coefficients  $\zeta$ ,  $\lambda$ , and  $\eta$  in Eq.'s (4.27)-(4.29) obtained in our formalism are independent of  $\theta$ . For this purpose, we introduce a new linearized collision operator

$$\begin{aligned} \mathcal{L}_{pq} &\equiv (p \cdot \theta_p) L_{pq} \\ &= -\frac{1}{2!} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p, p_1 | p_2, p_3) f_{p_1}^{\text{eq}} (\delta_{pq} + \delta_{p_1 q} - \delta_{p_2 q} - \delta_{p_3 q}), \end{aligned} \quad (5.1)$$

which is independent of  $\theta$ . We note that the inverse matrix of  $\mathcal{L}_{pq}$  reads

$$\mathcal{L}_{pq}^{-1} = L_{pq}^{-1} (q \cdot \theta_q)^{-1}. \quad (5.2)$$

Using  $\mathcal{L}_{pq}^{-1}$  and the  $\theta$ -independent microscopic representations of the dissipative currents  $(\Pi_p, J_p^\mu, \pi_p^{\mu\nu})$  in Eq.'s (4.13)-(4.15), we can represent the transport coefficients

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<sup>\*)</sup> In a previous short communication,<sup>36)</sup> we discussed the equation obtained with an erroneous setting  $\theta = -\pi/4$ . We note that this choice is not appropriate and the resulting equation should be abandoned since  $-\pi/4$  is out of range of  $\theta$  (4.7) that guarantees the positive definiteness of the inner product.

as

$$\zeta = -\frac{1}{T} \langle H, \mathcal{L}^{-1} H \rangle_{\text{eq}}, \quad (5.3)$$

$$\lambda = \frac{1}{3T^2} \langle J^\mu, \mathcal{L}^{-1} J_\mu \rangle_{\text{eq}}, \quad (5.4)$$

$$\eta = -\frac{1}{10T} \langle \pi^{\mu\nu}, \mathcal{L}^{-1} \pi_{\mu\nu} \rangle_{\text{eq}}, \quad (5.5)$$

where we have introduced the new inner product

$$\langle \varphi, \psi \rangle_{\text{eq}} \equiv \sum_p \frac{1}{p^0} f_p^{\text{eq}} \varphi_p \psi_p. \quad (5.6)$$

It is noted that this inner product is identically the thermal average, which is independent of  $\theta$ . Equations (5.3)-(5.5) show that  $\zeta$ ,  $\lambda$ , and  $\eta$  are  $\theta$ -invariant, and agree with the definitions used in the literature.

### 5.2. Absence of macroscopic-frame vector leading to Eckart's particle-frame equation

In §4, we have seen that the hydrodynamic equation in the particle frame does not take the form of Eckart, if the equation is to be derived from the relativistic Boltzmann equation. This fact might not be well known apart from some exceptions.<sup>15)</sup> For completeness and instructive purpose, we shall here give a proof that there can not exist a macroscopic-frame vector  $\mathbf{a}_p^\mu$  that leads to Eckart's constraints, (1.1), (1.2), and (1.3). It will be found in the course of the proof that the orthogonality condition between the  $P_0$  and  $Q_0$  space exactly corresponds to the phenomenological constraints imposed to the dissipative part of the energy-momentum tensor  $\delta T^{\mu\nu}$  and the particle current  $\delta N^\mu$  defined in Eq.'s (3.59) and (3.61).

We first note that the dissipative currents  $\delta J_{1\text{st}}^{\mu\alpha}$  given in Eq.(3.54) can be expressed as

$$\delta J_{1\text{st}}^{\mu\alpha} = \sum_p \frac{1}{p^0} p^\mu \varphi_{0p}^\alpha f_p^{\text{eq}} \bar{\phi}_p, \quad (5.7)$$

where  $\bar{\phi}_p$  is defined by

$$\bar{\phi}_p \equiv -[L^{-1} Q_0 f^{\text{eq}-1} F]_p = [L^{-1} \varphi_1^{\mu\alpha}]_p \bar{X}_{\mu\alpha}, \quad (5.8)$$

where  $\varphi_{1p}^{\mu\alpha}$  are the first-excited modes (3.67) and  $\bar{X}_{\mu\alpha}$  the thermodynamic forces defined by Eq.(3.63). Note that  $\bar{\phi}_p$  belongs to the  $Q_0$  space and is accordingly orthogonal to the five vectors  $\varphi_{0p}^\alpha$  belonging to the  $P_0$  space;

$$\langle \varphi_0^\alpha, \bar{\phi} \rangle = 0 \text{ for } \alpha = 0, 1, 2, 3, 4. \quad (5.9)$$

Here, the inner product is defined by Eq.(3.17) where  $\mathbf{a}_p^\mu$  enters in the form of  $(p \cdot \mathbf{a}_p)$ . We are going to show that these five identical equations exactly correspond to the ansatz's on  $\delta T^{\mu\nu}$  and  $\delta N^\mu$ ; see Eq.'s (3.59) and (3.61).



Firstly, let us take the case of the energy frame with the choice (A)  $\mathbf{a}_p^\mu = u^\mu$ . Then, Eq.(5.9) is reduced to

$$\sum_p \frac{1}{p^0} (p \cdot u) f_p^{\text{eq}} \varphi_{0p}^\alpha \bar{\phi}_p = 0, \quad (5.10)$$

which actually expresses a set of equations as follows

$$u_\nu \delta T^{\mu\nu} = 0, \quad (5.11)$$

$$u_\mu \delta N^\mu = 0. \quad (5.12)$$

Note that Eq.(5.11) implies the following two equations,

$$\delta e = u_\mu u_\nu \delta T^{\mu\nu} = 0, \quad (5.13)$$

$$Q_\rho = \Delta_{\rho\mu} u_\nu \delta T^{\mu\nu} = 0. \quad (5.14)$$

Thus, one can readily see that these equations coincide with Landau-Lifshitz's ansatz's, (1.1), (1.2), and (1.4).

Similarly, with the choice of (B)  $\mathbf{a}_p^\mu = m/(p \cdot u) u^\mu$ , we have

$$\sum_p \frac{1}{p^0} m f_p^{\text{eq}} \varphi_{0p}^\alpha \bar{\phi}_p = 0, \quad (5.15)$$

which means that

$$\delta N^\mu = 0, \quad (5.16)$$

$$\delta T^\mu_{\mu} = 0. \quad (5.17)$$

Here, we have used the on-shell condition  $m^2 = p^\mu p_\mu$ . Equation (5.16) is equivalent to the set of equations

$$\delta n = u_\mu \delta N^\mu = 0, \quad (5.18)$$

$$\nu_\nu = \Delta_{\nu\mu} \delta N^\mu = 0. \quad (5.19)$$

Thus, one sees that Stewart's ansatz's, (1.5), (1.2), and (1.3), are derived in the particle frame.

It is now easy to see that there exists no macroscopic-frame vector  $\mathbf{a}_p^\mu$  leading to Eckart's ansatz's given by Eq.'s (1.1), (1.2), and (1.3), simultaneously. Indeed, in order to lead to the ansatz's (1.2) and (1.3) on  $\delta N^\mu$ , Eq.(5.9) with  $\alpha = \mu$  requires that  $(p \cdot \mathbf{a}_p)$  is independent of  $p^\mu$ , i.e.,

$$(p \cdot \mathbf{a}_p) = \text{const.} \quad (5.20)$$

On the other hand, in order to lead to the ansatz (1.1) on  $\delta T^{\mu\nu}$ , Eq.(5.9) with  $\alpha = 4$  must lead to

$$(p \cdot \mathbf{a}_p) = \text{const.} \times (p \cdot u)^2, \quad (5.21)$$

which is in contradiction with Eq.(5.20). Thus, we conclude that there exists no  $\mathbf{a}_p^\mu$  leading to Eckart's ansatz's (1.1)-(1.3) simultaneously.

In short, the phenomenological ansatz's on the dissipative parts of the energy-momentum tensor  $\delta T^{\mu\nu}$  and the particle current  $\delta N^\mu$  have a definite correspondence to the orthogonality conditions posed to the dissipative part of the distribution function as a solution to the underlying Boltzmann equation. In particular, whenever the particle frame is taken where the particle flow is constructed so as to have no dissipative part, the dissipative part of the energy-momentum tensor must satisfy Eq.(1.5) but not Eq.(1.1). This fact implies that the Eckart equation can not be the hydrodynamic equation in the particle frame as the slow (or infrared) dynamics of the relativistic Boltzmann equation, and hence nor have microscopic foundation.

### 5.3. *Stability analysis of steady solution of relativistic hydrodynamic equation*

In this subsection, we shall give a general proof that steady solutions of our generic relativistic dissipative hydrodynamic equation (3.68) are stable against a small perturbation, on account of the positive definiteness of the inner product (3.18). Here, a steady solution means that it describes a system having a finite homogeneous flow with a constant temperature and a constant chemical potential, as follows,

$$T(\sigma; \tau) = T_0, \quad (5.22)$$

$$\mu(\sigma; \tau) = \mu_0, \quad (5.23)$$

$$u_\mu(\sigma; \tau) = u_{0\mu}, \quad (5.24)$$

where  $T_0$ ,  $\mu_0$ , and  $u_{0\mu}$  are constant. We note that the steady states include the thermal equilibrium state as a special case. We remark that such a stability of our equation was demonstrated in a previous paper by the present authors<sup>37)</sup> by a numerical calculation for a rarefied gas with some specific models for the cross section in the collision integral. The general proof of the stability of our equation is given for the first time in the present paper.

To show the stability of the steady solution, we apply the so-called linear stability analysis to the relativistic dissipative hydrodynamic equation (3.68). We expand  $T$ ,  $\mu$  and  $u_\mu$  around the steady solution as follows:

$$T(\sigma; \tau) = T_0 + \delta T(\sigma; \tau), \quad (5.25)$$

$$\mu(\sigma; \tau) = \mu_0 + \delta\mu(\sigma; \tau), \quad (5.26)$$

$$u_\mu(\sigma; \tau) = u_{0\mu} + \delta u_\mu(\sigma; \tau). \quad (5.27)$$

We assume that the higher-order terms than the second order with respect to  $\delta T$ ,  $\delta\mu$ , and  $\delta u_\mu$  can be neglected since these quantities are small. For convenience, we introduce the new variables  $\delta X_\alpha$  composed of  $\delta T$ ,  $\delta\mu$ , and  $\delta u_\mu$  by

$$\delta X_\alpha \equiv \begin{cases} -\delta(u_\mu/T) = -\delta u_\mu/T_0 + \delta T u_{0\mu}/T_0^2 & \text{for } \alpha = \mu, \\ m^{-1} \delta(\mu/T) = m^{-1} (\delta\mu/T_0 - \delta T \mu_0/T_0^2) & \text{for } \alpha = 4. \end{cases} \quad (5.28)$$

Substituting Eq.(5.28) into Eq.(3.68), we obtain the linearized equation governing

$\delta X_\alpha$  as

$$\begin{aligned} & \left( \langle \varphi_0^\alpha, \varphi_0^\beta \rangle + \langle \varphi_0^\alpha, L^{-1} \varphi_1^{\nu\beta} \rangle \nabla_\nu \right) \frac{\partial}{\partial \tau} \delta X_\beta \\ & + \left( \langle \tilde{\varphi}_1^{\mu\alpha}, \varphi_0^\beta \rangle \nabla_\mu + \langle \tilde{\varphi}_1^{\mu\alpha}, L^{-1} \varphi_1^{\nu\beta} \rangle \nabla_\mu \nabla_\nu \right) \delta X_\beta = 0, \end{aligned} \quad (5.29)$$

with  $\tilde{\varphi}_{1p}^{\mu\alpha} = p^\mu \varphi_{0p}^\alpha / (p \cdot \mathbf{a}_p)$  defined in Eq.(3.55). Here, we have used the following simple relations

$$\delta(f_p^{\text{eq}}) = f_p^{\text{eq}} \varphi_{0p}^\alpha \delta X_\alpha, \quad (5.30)$$

$$\delta(\tilde{X}_{\mu\alpha}) = \nabla_\mu \delta X_\alpha. \quad (5.31)$$

We note that all of the coefficients in Eq.(5.29) take a value of the steady solution  $(T, \mu, u_\mu) = (T_0, \mu_0, u_{0\mu})$ . With use of the orthogonality condition between the  $P_0$  and  $Q_0$  spaces and the definitions of  $\eta_0^{\alpha\beta}$  and  $\eta_1^{\mu\alpha\nu\beta}$  in Eq.'s (3.26) and (3.66), respectively, we can reduce Eq.(5.29) to

$$A^{\alpha,\beta} \frac{\partial}{\partial \tau} \delta X_\beta + B^{\alpha,\beta} \delta X_\beta = 0, \quad (5.32)$$

where  $A^{\alpha,\beta}$  and  $B^{\alpha,\beta}$  are defined by

$$A^{\alpha,\beta} \equiv \eta_0^{\alpha\beta}, \quad (5.33)$$

$$B^{\alpha,\beta} \equiv \langle \tilde{\varphi}_1^{\mu\alpha}, \varphi_0^\beta \rangle \nabla_\mu + \eta_1^{\mu\alpha\nu\beta} \nabla_\mu \nabla_\nu. \quad (5.34)$$

We convert Eq.(5.32) into the algebraic equation, using the Fourier and Laplace transformations with respect to the spatial variable  $\sigma^\mu$  and the temporal variable  $\tau$ , respectively: By inserting

$$\delta X_\alpha(\sigma; \tau) = \delta \tilde{X}_\alpha(k; \Lambda) e^{ik \cdot \sigma - \Lambda \tau}, \quad (5.35)$$

into Eq.(5.32), we have

$$(\Lambda A^{\alpha,\beta} - \tilde{B}^{\alpha,\beta}) \delta \tilde{X}_\beta = 0, \quad (5.36)$$

where  $\tilde{B}^{\alpha,\beta}$  is defined by

$$\tilde{B}^{\alpha,\beta} \equiv i \langle \tilde{\varphi}_1^{\mu\alpha}, \varphi_0^\beta \rangle k_\mu - \eta_1^{\mu\alpha\nu\beta} k_\mu k_\nu. \quad (5.37)$$

In the rest of this section, we use the matrix representation when no misunderstanding is expected. Since we are interested in a nonvanishing solution  $\delta \tilde{X} \neq 0$ , we impose

$$\det(\Lambda A - \tilde{B}) = 0, \quad (5.38)$$

which leads to the dispersion relation

$$\Lambda = \Lambda(k). \quad (5.39)$$

The stability of the steady solution (5.22)-(5.24) against a small perturbation means that  $\delta X$  will not increase under time evolution. Therefore, our next task is to show that the real part of  $\Lambda(k)$  is non-negative for any  $k^\mu$ .

Now we first note that the matrix  $A$  is a real symmetric and positive-definite matrix:

$$\begin{aligned} w_\alpha A^{\alpha,\beta} w_\beta &= \langle w_\alpha \varphi_0^\alpha, w_\beta \varphi_0^\beta \rangle \\ &= \langle \varphi, \varphi \rangle > 0 \text{ for } w_\alpha \neq 0, \end{aligned} \quad (5.40)$$

with  $\varphi_p \equiv w_\alpha \varphi_{0p}^\alpha$ . Here, we have used the positive definiteness of the inner product (3.18). Equation (5.40) means that the inverse matrix  $A^{-1}$  exists, and  $A^{-1}$  is also a real symmetric positive-definite matrix. Thus, using the Cholesky decomposition, we can represent  $A^{-1}$  as

$$A^{-1} = {}^t U U, \quad (5.41)$$

where  $U$  denotes a real matrix and  ${}^t U$  a transposed matrix of  $U$ . Substituting Eq.(5.41) into Eq.(5.38), we have

$$\det(\Lambda - U \tilde{B} {}^t U) = 0. \quad (5.42)$$

It is noted that  $\Lambda(k)$  is an eigen value of  $U \tilde{B} {}^t U$ .

We notice the following theorem: The real part of the eigen value of a complex matrix  $C$  is non-negative when the hermite matrix  $\text{Re}(C) \equiv (C + C^\dagger)/2$  is semi-positive definite. Applying this theorem to the present case, we find that the real part of  $\Lambda(k)$  becomes non-negative for any  $k^\mu$  when  $\text{Re}(U \tilde{B} {}^t U)$  is a semi-positive definite matrix. In fact, we can show that  $\text{Re}(U \tilde{B} {}^t U)$  is semi-positive definite as follows:

$$\begin{aligned} w_\alpha [\text{Re}(U \tilde{B} {}^t U)]^{\alpha,\beta} w_\beta &= w_\alpha [U \text{Re}(\tilde{B}) {}^t U]^{\alpha,\beta} w_\beta \\ &= [w U]_\alpha [\text{Re}(\tilde{B})]^{\alpha,\beta} [w U]_\beta \\ &= -[w U]_\alpha \eta_1^{\mu\alpha\nu\beta} k_\mu k_\nu [w U]_\beta \\ &= -\langle k_\mu [w U]_\alpha \varphi_1^{\mu\alpha}, L^{-1} k_\nu [w U]_\beta \varphi_1^{\nu\beta} \rangle \\ &= -\langle \psi, L^{-1} \psi \rangle \geq 0 \text{ for } w_\alpha \neq 0, \end{aligned} \quad (5.43)$$

with  $\psi_p \equiv k_\mu [w U]_\alpha \varphi_{1p}^{\mu\alpha}$ . Therefore, we conclude that the steady solution in Eq.'s (5.22)-(5.24) is stable against a small perturbation.

We now see that the relativistic dissipative hydrodynamic equation (3.68) obtained by the RG method has a stable steady solution in a definite way. It is noteworthy that this property is kept in the several equations derived in the setting of  $\mathbf{a}_p^\mu = \theta_p^\mu$ , i.e., the energy-frame equation in Eq.'s (4.68) and (4.69), the particle-frame equations in Eq.'s (4.83) and (4.84), and so on. We stress that this is for the first time that the relativistic dissipative hydrodynamic equation in the particle frame has been obtained whose steady solution is stable <sup>\*)</sup>.

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<sup>\*)</sup> In the previous rapid communication,<sup>37)</sup> we discussed the stability only of the thermal-equilibrium solution, i.e.,  $u_{0\mu} = (1, 0, 0, 0)$ , with use of a rarefied-gas approximation, where a differential cross section in the collision integral is treated as a constant.

## §6. Summary and Concluding Remarks

In this paper, we have given a full and detailed account of the derivation of the first-order relativistic dissipative hydrodynamic equations as a reduction of dynamics from the relativistic Boltzmann equation with no heuristic assumptions, such as the so-called conditions of fit used in the standard methods. This was made possible by adopting a powerful reduction theory of dynamics, i.e., the renormalization-group method and by introducing the macroscopic-frame vector which defines the macroscopic local rest frame; thereby we successfully have a coarse-grained and Lorentz-covariant equations in a generic frame.

The five hydrodynamic modes are naturally identified with the same number of the zero modes of the linearized collision operator. The excited modes which are to modify the local equilibrium distribution function and give the dissipative terms are defined so that they are precisely orthogonal to the zero modes with a properly defined inner product for the distribution functions. It is worth emphasizing that the dissipative terms are constructed without recourse to the so-called conditions of fit which are imposed to the dissipative terms in an ad-hoc way in the standard methods. In our method, the validity of the conditions of fit is checked as a property of the derived equations. On the basis of the nice properties of the properly defined inner product and the very nature of the hydrodynamic modes as the zero modes of the linearized collision operator, we have shown that the so-called Burnett term does not affect the hydrodynamic equations.

Our hydrodynamic equation with the dissipative currents given by Eq.(4-44), which still contains the macroscopic-frame vector, is a master equation for a generic local rest frame, and derives hydrodynamic equations in various local rest frames with a specific choice of the macroscopic-frame vector. We have shown that our energy-momentum tensor and particle current in the energy frame, given by Eq.'s (4-68) and (4-69), respectively, coincide with those proposed by Landau and Lifshitz, but those in the particle frame, given by Eq.'s (4-83) and (4-84), are slightly different from those given by Eckart and by Stewart. Our generic equation is also reduced to a novel relativistic hydrodynamic equation with the energy-momentum tensor and the particle current given by Eq.'s (4-98) and (4-98), respectively, where the bulk pressure (viscosity) term is absent.

Furthermore, we have proved that the derived equation in a generic local rest frame has a stable equilibrium state owing to the positive definiteness of the inner product, although such a generic stability was suggested by a numerical calculation for some specific parameter sets in a previous paper by the present authors. It is worth emphasizing that all of our equations have a stable equilibrium state even in the particle frame in contrast to the Eckart and Stewart equations; note that such a drawback of the equation that the thermal equilibrium state can be unstable is taken over to some causal equations including the Israel-Stewart equation.

In conformity with the above fact, we have proved that the Eckart equation can not be compatible with the underlying relativistic Boltzmann equation. This proof is based on the following significant observation that the orthogonality condition of the excited modes to the zero modes coincides with the ansatz's posed on the

dissipative parts of the energy-momentum tensor and the particle current in the phenomenological equations.

Thus, we have arrived at a sound starting point for attacking the last problem, i.e., constructing a causal relativistic hydrodynamic equation: We can expect that the present method applies to derive a causal equation on a sound basis, without recourse to any ansatz's as is done in other method such as the Maxwell-Grad moment method.<sup>18),20),26)</sup> We have found<sup>21)</sup> that a non-trivial extension of the  $P_0$  space in our formalism gives a causal equation, whose low-frequency limit is identical to the equation derived in this paper. This extension to derive causal equations will be reported in the forthcoming paper,<sup>21)</sup> where a correct moment method will be also proposed.

We stress that all of the equations derived in this paper are consistent with the underlying kinetic equation, so the equations and also the method developed here may be useful for the analysis of the system where the proper dynamics describing the system changes from the hydrodynamic to kinetic regime, as in the system near the freeze-out region in the RHIC phenomenology.<sup>42)–44)</sup>

Due to the absence of the problematic Burnett term in our relativistic equations, we expect that applying the RG method to the non-relativistic Boltzmann equation leads to the Navier-Stokes equation without the Burnett term, whose existence is known to be inevitable<sup>39)</sup> when the Chapman-Enskog expansion method is applied.

Finally, we emphasize that our method itself has a universal nature and can be applied to derive a slow dynamics from kinetic equations other than the simple Boltzmann equation. For example, it is successfully applicable to derive the dissipative hydrodynamic equation for the multi-component system in the energy frame from the multi-component Boltzmann equation.<sup>18)</sup> Furthermore, it would be interesting to apply the present method for extracting a Lorentz-covariant hydrodynamics of strongly interacting systems out of the equilibrium from the Kadanoff-Baym quantum-transport equation<sup>45)</sup> for the non-equilibrium many-body Green's function.

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## Appendix A

——— *Most general form of phenomenological relativistic dissipative hydrodynamic equations with dissipative internal energy  $\delta e$ , pressure  $\delta p$ , and density  $\delta n$*  ———

The most general form of the dissipative parts of the energy-momentum tensor and particle current takes the following forms,

$$\delta T^{\mu\nu} = \delta e u^\mu u^\nu - \delta p \Delta^{\mu\nu} + Q^\mu u^\nu + Q^\nu u^\mu + \Pi^{\mu\nu}, \quad (\text{A}\cdot 1)$$

$$\delta N^\mu = \delta n u^\mu + \nu^\mu, \quad (\text{A}\cdot 2)$$

respectively. Note that we have made it explicit that the dissipations may cause corrections to the internal energy  $\delta e$ , the pressure  $\delta p$  and the particle-number density  $\delta n$ , although only  $\delta p$  is usually considered (as the bulk pressure term) in the literature. We remark that  $\delta e$  and  $\delta p$  are related to the dissipative part of the energy-momentum tensor, as follows,

$$u_\mu \delta T^{\mu\nu} u_\nu = \delta e, \quad (\text{A}\cdot 3)$$

$$\delta T^\mu_\mu = \delta e - 3 \delta p. \quad (\text{A}\cdot 4)$$

In the present parametrization,  $e + \delta e$ ,  $p + \delta p$ , and  $n + \delta n$  are the internal energy, pressure, and particle-number density in the dissipative system, with  $e = e(T, \mu)$ ,  $p = p(T, \mu)$ , and  $n = n(T, \mu)$  being the corresponding quantities in the local equilibrium state with the temperature  $T$  and the chemical potential  $\mu$ . In the tensor decomposition,

$$Q^\mu \equiv \Delta^{\mu\nu} u^\rho T_{\nu\rho}, \quad (\text{A}\cdot 5)$$

$$\nu^\mu \equiv \Delta^{\mu\nu} N_\nu, \quad (\text{A}\cdot 6)$$

$$\Pi^{\mu\nu} \equiv \Delta^{\mu\nu\rho\sigma} T_{\rho\sigma}, \quad (\text{A}\cdot 7)$$

where  $\Delta^{\mu\nu\rho\sigma}$  is the space-like, symmetric and traceless tensor defined in Eq.(4.23).

Owing to the properties of the dissipative parts,  $Q^\mu u_\mu = 0$ ,  $\nu^\mu u_\mu = 0$ ,  $\Pi^{\mu\nu} = \Pi^{\nu\mu}$ , and  $u_\mu \Pi^{\mu\nu} = \Pi^\mu_\mu = 0$ , the total number of independent components of  $Q^\mu$ ,  $\nu^\mu$ , and  $\Pi^{\mu\nu}$  is eleven. Since  $T^{\mu\nu}$  and  $N^\mu$  have fourteen components in total,  $\delta e$ ,  $\delta p$ , and  $\delta n$  can not be independent of each other, but the number of independent component is one other than  $T$  and  $\mu$ . We take  $\delta p = \Pi$  as the independent component as a natural choice, then  $\delta e$  and  $\delta n$  may be expressed as  $\delta e = f_e \Pi$  and  $\delta n = f_n \Pi$ , where  $f_e$  and  $f_n$  are functions of  $T$  and  $\mu$ ;  $f_e = f_e(T, \mu)$  and  $f_n = f_n(T, \mu)$ . Here, we have assumed that the dissipative order of  $\delta e$  and  $\delta n$  is the same as that of  $\delta p$  at most. We remark that in terms of  $f_e$ ,

$$u_\mu \delta T^{\mu\nu} u_\nu = f_e \Pi, \quad (\text{A}\cdot 8)$$

$$\delta T^\mu_\mu = (f_e - 3) \Pi. \quad (\text{A}\cdot 9)$$

Now we shall show that the usual phenomenological derivation of the hydrodynamic equations allows the existence of  $\delta e$  and  $\delta n$ , i.e., finite values of  $f_e$  and  $f_n$ , in

the relativistic dissipative hydrodynamic equations. We emphasize that it can not be excluded on the general ground that  $f_e$  and  $f_n$  may have finite values, although the phenomenological theory can not determine their parametric forms as functions of  $T$  and  $\mu$ . Although the first-order equation is considered in this article, the same argument equally applies to the second-order equations.

Now the entropy current is given by

$$T S^\mu = p u^\mu + u_\nu T^{\mu\nu} - \mu N^\mu. \quad (\text{A}\cdot 10)$$

The second law of thermodynamics reads  $\partial_\mu S^\mu \geq 0$ .

The divergence of  $S^\mu$  is found to take the form

$$\begin{aligned} \partial_\mu S^\mu = & \Pi \left[ f_e D \frac{1}{T} - \frac{1}{T} \nabla^\mu u_\mu - f_n D \frac{\mu}{T} \right] + Q^\mu \left[ \frac{1}{T} D u_\mu + \nabla_\mu \frac{1}{T} \right] \\ & - \nu^\mu \nabla_\mu \frac{\mu}{T} + \Pi^{\mu\nu} \frac{1}{T} \nabla_\mu u_\nu, \end{aligned} \quad (\text{A}\cdot 11)$$

where  $D \equiv u^\mu \partial_\mu$  and  $\nabla^\mu \equiv \Delta^{\mu\nu} \partial_\nu$ . Here, we have used the hydrodynamic equation and the first law of thermodynamics,

$$D(p/T) + e D(1/T) - n D(\mu/T) = 0. \quad (\text{A}\cdot 12)$$

In the particle frame where  $\nu^\mu = 0$ , Eq.(A.11) is reduced to

$$\begin{aligned} \partial_\mu S^\mu = & \Pi \left[ f_e D \frac{1}{T} - \frac{1}{T} \nabla^\mu u_\mu - f_n D \frac{\mu}{T} \right] + Q^\mu \left[ \frac{1}{T} D u_\mu + \nabla_\mu \frac{1}{T} \right] \\ & + \Pi^{\mu\nu} \frac{1}{T} \nabla_\mu u_\nu. \end{aligned} \quad (\text{A}\cdot 13)$$

For assuring the second law of thermodynamics, one should make the right-hand side semi-positive definite. A natural choice is the following constitutive equations,

$$\Pi = \zeta T \left[ f_e D \frac{1}{T} - \frac{1}{T} \nabla^\mu u_\mu - f_n D \frac{\mu}{T} \right], \quad (\text{A}\cdot 14)$$

$$Q^\mu = -\lambda T^2 \left[ \frac{1}{T} D u^\mu + \nabla^\mu \frac{1}{T} \right], \quad (\text{A}\cdot 15)$$

$$\Pi^{\mu\nu} = 2\eta \Delta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma, \quad (\text{A}\cdot 16)$$

with  $\zeta$ ,  $\lambda$ , and  $\eta$  being the bulk viscosity, heat conductivity, and shear viscosity, respectively. Indeed, the above constitutive equations lead to

$$\partial_\mu S^\mu = \frac{\Pi^2}{\zeta T} - \frac{Q^\mu Q_\mu}{\lambda T^2} + \frac{\Pi^{\mu\nu} \Pi_{\mu\nu}}{2\eta T} \geq 0. \quad (\text{A}\cdot 17)$$

Thus, one sees that the relativistic dissipative hydrodynamic equations with finite  $f_e$  and  $f_n$ , or equivalently with finite  $\delta e$  and  $\delta n$ , is compatible with the second



law of thermodynamics. With a special choice,  $f_e = f_n = 0$ , Eq.'s (A.14)-(A.16) lead to the constitutive equations proposed by Eckart that are commonly used.

In the case of the energy frame where  $Q^\mu = 0$ , a similar argument leads to Eq.'s (A.14), (A.16), and

$$\nu^\mu = \lambda \hat{h}^{-2} \nabla^\mu \frac{\mu}{T}, \quad (\text{A.18})$$

with  $\hat{h} = (e + p)/nT$  being the reduced enthalpy per particle. We remark that these equations are reduced to the constitutive equations by Landau and Lifshitz only when one can set  $f_e = f_n = 0$ .

The phenomenological theory can not proceed further to specify  $f_e$  and  $f_n$ ; the values of  $f_e$  and  $f_n$  can be determined only from a microscopic theory. In the present work, we show that the microscopic theory leads to  $f_e = f_n = 0$  in the energy frame, while  $f_e = 3$  and  $f_n = 0$  in the particle frame, implying that  $\delta T^\mu_\mu = 0$  but  $u_\mu \delta T^{\mu\nu} u_\nu = 3\Pi \neq 0$ .

## Appendix B

—— *Brief account of the renormalization-group method with an example* ——

In this Appendix, we first show how the renormalization-group (RG) method<sup>28)-33)</sup> works as a reduction theory of the dynamics, adopting van der Pol equation with a limit cycle, as an example. Then, we briefly present a foundation of the RG method. A detailed account of the RG method including its foundation may be seen in Ref.'s 29), 30), 32), 33) or review articles 35), 46).

### B.1. RG method applied to van der Pol equation

Let us take the van der Pol equation which admits a limit cycle:

$$\ddot{x} + x = \epsilon(1 - x^2)\dot{x}, \quad (\text{B.1})$$

where  $\epsilon$  is supposed to be small.

Let us solve the same problem by the RG method. Let  $\tilde{x}(t; t_0)$  be a local solution around  $t \sim \forall t_0$ , and represent it as a perturbation series;  $\tilde{x}(t; t_0) = \tilde{x}_0(t; t_0) + \epsilon \tilde{x}_1(t; t_0) + \epsilon^2 \tilde{x}_2(t; t_0) + \dots$ . In the RG method, the initial value  $W(t_0)$  matters: We suppose that an exact solution is given by  $x(t)$  and the initial value of  $\tilde{x}(t; t_0)$  at  $t = t_0$  is set up to be  $x(t_0)$ ; i.e.,  $W(t_0) \equiv \tilde{x}(t_0; t_0) = x(t_0)$ . The initial value as the exact solution should be also expanded as  $W(t_0) = W_0(t_0) + \epsilon W_1(t_0) + \epsilon^2 W_2(t_0) + \dots$ . The RG method is actually the method to obtain the initial value  $W(t)$  as an exact solution or approximate solution valid in a global domain asymptotically.

The zeroth-order equation reads

$$\mathcal{L}\tilde{x}_0 \equiv \left[ \frac{d^2}{dt^2} + 1 \right] \tilde{x}_0 = 0, \quad (\text{B.2})$$

the solution to which can be written as

$$\tilde{x}_0(t; t_0) = A(t_0) \cos(t + \theta(t_0)). \quad (\text{B.3})$$

Here, we have made it explicit that the integral constants  $A$  and  $\theta$  may depend on the initial time  $t_0$ . The equation for  $\tilde{x}_1$  reads

$$\mathcal{L}\tilde{x}_1 = -A \left(1 - \frac{A^2}{4}\right) \sin \phi(t) + \frac{A^3}{4} \sin 3\phi(t), \quad (\text{B}\cdot 4)$$

with  $\phi(t) = t + \theta_0(t_0)$ . Notice that the first term in the right-hand side is a zero mode of the linear operator  $\mathcal{L}$  appearing in the left-hand side. Thus, the special solution to this equation necessarily contains a secular term which is given by  $t$  times a zero mode of  $\mathcal{L}$ . Since we have supposed that the initial value at  $t = t_0$  is on an exact solution, the corrections from the zeroth-order solution should be as small as possible. This condition is realized by setting the secular terms appearing in the higher orders vanish at  $t = t_0$ , which is possible because we can add freely zero mode solutions to a special solution. Thus, the first-order solution is uniquely written as

$$\tilde{x}_1(t; t_0) = (t - t_0) \frac{A}{2} \left(1 - \frac{A^2}{4}\right) \sin \phi(t) - \frac{A^3}{32} \sin 3\phi(t). \quad (\text{B}\cdot 5)$$

Notice that the secular term surely vanishes at  $t = t_0$  in Eq.(B.5), implying that its initial value at  $t = t_0$  reads

$$\tilde{x}_1(t_0; t_0) = -\frac{A^3(t_0)}{32} \sin 3\phi(t_0). \quad (\text{B}\cdot 6)$$

The perturbative solution up to this order reads  $\tilde{x} = \tilde{x}_0 + \epsilon \tilde{x}_1$ , which becomes, however, invalid when  $|t - t_0| \rightarrow \text{large}$ , because of the secular term.

Now notice that the function  $\tilde{x}(t; t_0)$  corresponds to a curve drawn in the  $(t, x)$  plane for each  $t_0$ ; in other words, we have a family of curves represented by  $\tilde{x}(t; t_0)$  in the  $(t, x)$  plane, a member of which is parametrized by  $t_0$ . An important observation is that each curve is close to the exact solution in the neighborhood of  $t = t_0$ . Thus, an idea is that the envelope curve of the family of curves should give a global solution. The envelope curve can be constructed by solving the following equation

$$\left. \frac{d\tilde{x}}{dt} \right|_{t_0=t} = 0, \quad (\text{B}\cdot 7)$$

which leads to the following equations for  $A(t)$  and  $\phi(t)$ ,

$$\dot{A} = \epsilon \frac{A}{2} \left(1 - \frac{A^2}{4}\right), \quad (\text{B}\cdot 8)$$

$$\dot{\phi} = 1. \quad (\text{B}\cdot 9)$$

These equation are readily solved, and one sees that as  $t \rightarrow \infty$ ,  $A(t) \rightarrow 2$  asymptotically, meaning the existence of a limit cycle with a radius 2.

The resultant envelope function as a global solution is given by

$$x_E(t) \equiv \tilde{x}(t; t) = W(t) = A(t) \cos(t + \theta_0) - \epsilon \frac{A^3(t)}{32} \sin(3t + 3\theta_0), \quad (\text{B}\cdot 10)$$

with  $A(t)$  being the solution of Eq.(B·8). Thus, we have succeeded in not only obtaining the asymptotic solution as whole but also extracting the slow variables  $A(t)$  and  $\phi(t)$  explicitly and their governing equations.

One now sees that when there exist zero modes of the unperturbed operator, the higher-order corrections may cause secular terms, which are renormalized into the integral constants in the zeroth-order solution by the RG/envelope equation (B·7), and thereby the would-be integral constants are lifted to dynamical but slow variables.

In the derivation of hydrodynamic equations from the kinetic equation, the would-be integral constants corresponding to  $A$  and  $\phi$  are the temperature  $T$ , the chemical potential  $\mu$ , and the flow velocity  $u^\mu$  ( $u^\mu u_\mu = 1$ ) characterizing the local equilibrium state, which are to be lifted to the slow dynamical variables and their governing equations are identically the hydrodynamic equation.

## B.2. Foundation of RG method

Now we present a foundation to the RG method using a Wilsonian equation<sup>47)</sup> or flow equation by Wegener.<sup>48)</sup>

Let us take the following  $n$ -dimensional equation;

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}, t), \quad (\text{B} \cdot 11)$$

where  $n$  may be infinity. Let  $\mathbf{X}(t) = \mathbf{W}(t)$  be an yet unknown exact solution to Eq.(B·11), and we try to solve the equation with the initial condition at  $t = \forall t_0$ ;

$$\mathbf{X}(t = t_0) = \mathbf{W}(t_0). \quad (\text{B} \cdot 12)$$

Then, the solution may be written as  $\mathbf{X}(t; t_0, \mathbf{W}(t_0))$ .

Now the basis of the RG method lies in the fact that  $\mathbf{W}(t_0)$  can be determined on the basis of a simple fact of differential equations. We notice that when the initial point is shifted to  $t'_0$ , the resultant solution should be the same as long as  $\mathbf{W}(t)$  is an exact solution, i.e.,

$$\mathbf{X}(t; t_0, \mathbf{W}(t_0)) = \mathbf{X}(t; t'_0, \mathbf{W}(t'_0)). \quad (\text{B} \cdot 13)$$

Taking the limit  $t'_0 \rightarrow t_0$ , we have

$$\frac{d\mathbf{X}}{dt_0} = \frac{\partial \mathbf{X}}{\partial t_0} + \frac{\partial \mathbf{X}}{\partial \mathbf{W}} \frac{d\mathbf{W}}{dt_0} = \mathbf{0}. \quad (\text{B} \cdot 14)$$

This equation gives an evolution equation or the flow equation of the initial value  $\mathbf{W}(t_0)$ . This equation has the same form as and corresponds to the non-perturbative RG equations (flow equations) by Wilson,<sup>47)</sup> Wegner-Houghton,<sup>48)</sup> and so on, in quantum field theory and statistical physics: The 'initial time'  $t_0$  corresponds to (the logarithm of) the renormalization point. We emphasize that the equation (B·14) is exact; we have no recourse to any perturbation theory so far.

The problem is to construct the seed of the RG equation (B·14), i.e.,  $\mathbf{X}(t; t_0, \mathbf{W}(t_0))$ . For that, let us take the perturbation theory. In this case,  $\mathbf{X}(t; t_0, \mathbf{W}(t_0))$  and

$\mathbf{X}(t; t'_0, \mathbf{W}(t'_0))$  may be valid only for  $t \sim t_0$  and  $t \sim t'_0$ , which condition is naturally satisfied when  $t_0 < t < t'_0$  (or  $t'_0 < t < t_0$ ) because the limit  $t'_0 \rightarrow t_0$  is taken eventually. Thus, when a perturbative expansion is employed for constructing  $\mathbf{X}(t; t_0, \mathbf{W}(t_0))$ , it is necessary to demand

$$\left. \frac{d\mathbf{X}}{dt_0} \right|_{t_0=t} = \left. \frac{\partial \mathbf{X}}{\partial t_0} \right|_{t_0=t} + \left. \frac{\partial \mathbf{X}}{\partial \mathbf{W}} \frac{d\mathbf{W}}{dt_0} \right|_{t_0=t} = \mathbf{0}, \quad (\text{B}\cdot 15)$$

which automatically implies the condition that  $t_0 = t$ .

We have already suggested an interpretation that this equation can be identified as a condition to construct an envelope of the curves represented by the unperturbed solutions with different initial times  $t_0$ 's.<sup>29)</sup> When  $t_0$  is varied,  $\mathbf{X}(t; t_0, \mathbf{W}(t_0))$  gives a family of curves with  $t_0$  being a parameter characterizing curves. Then, Eq.(B.15) is a condition to construct the envelope of the family of curves which are valid only locally around  $t \sim t_0$ . The envelope is given by  $\mathbf{X}(t; t_0 = t) = \mathbf{W}(t)$ , i.e, the initial value.

Now we shall show that  $\mathbf{X}(t; t_0 = t) = \mathbf{W}(t)$  satisfies the original equation (B.11) in a global domain up to the order with which  $\mathbf{X}(t; t_0)$  satisfies around  $t \sim t_0$ . Let  $\mathbf{X}(t; t_0)$  is an approximate solution to Eq.(B.11) around  $t \sim t_0$ ;

$$\frac{d\mathbf{X}(t; t_0)}{dt} \simeq \mathbf{F}(\mathbf{X}(t; t_0), t). \quad (\text{B}\cdot 16)$$

Then, we have

$$\begin{aligned} \frac{d\mathbf{W}(t)}{dt} &= \left. \frac{\partial \mathbf{X}(t; t_0)}{\partial t} \right|_{t_0=t} + \left. \frac{\partial \mathbf{X}(t; t_0)}{\partial t_0} \right|_{t_0=t} \\ &= \left. \frac{\partial \mathbf{X}(t; t_0)}{\partial t} \right|_{t_0=t} \\ &\simeq \mathbf{F}(\mathbf{X}(t; t_0), t)|_{t_0=t}, \\ &= \mathbf{F}(\mathbf{W}(t), t), \end{aligned} \quad (\text{B}\cdot 17)$$

on account of Eq.(B.15). This proves the above statement.

### Appendix C

—— Derivation of first-excited modes with a generic macroscopic-frame Vector

In this Appendix, we explicitly derive the first-excited modes  $\varphi_{1p}^{\mu\alpha}$  in the generic local rest frame with the macroscopic-frame vector being specified by Eq.(4.6), i.e.,  $\mathbf{a}_p^\mu = \theta_p^\mu = [((p \cdot u) \cos \theta + m \sin \theta)/(p \cdot u)] u^\mu$ .

For convenience, we introduce a dimensionless quantity<sup>18)</sup> dependent on  $z = m/T$ ,

$$\begin{aligned} a_\ell &\equiv \frac{1}{n T^{\ell-1}} \sum_p \frac{1}{p^0} f_p^{\text{eq}} (p \cdot u)^\ell \\ &= \frac{1}{z^2 K_2(z)} \int_z^\infty d\tau (\tau^2 - z^2)^{1/2} \tau^\ell e^{-\tau}. \end{aligned} \quad (\text{C}\cdot 1)$$

This quantity for  $\ell = 0, 1, 2, 3, 4, 5$  reads

$$a_0 = z^{-2} (\hat{h} - 4), \quad (\text{C}\cdot 2)$$

$$a_1 = 1, \quad (\text{C}\cdot 3)$$

$$a_2 = \hat{h} - 1, \quad (\text{C}\cdot 4)$$

$$a_3 = 3 \hat{h} + z^2, \quad (\text{C}\cdot 5)$$

$$a_4 = (15 + z^2) \hat{h} + 2 z^2, \quad (\text{C}\cdot 6)$$

$$a_5 = 6 (15 + z^2) \hat{h} + z^2 (15 + z^2), \quad (\text{C}\cdot 7)$$

respectively. Here,  $\hat{h}$  is defined in Eq.(4.21) and  $K_\ell(z)$  denotes the modified Bessel function,

$$K_\ell(z) = \frac{2^\ell \ell!}{(2\ell)!} z^{-\ell} \int_z^\infty d\tau (\tau^2 - z^2)^{\ell-1/2} e^{-\tau}, \quad (\text{C}\cdot 8)$$

which satisfies the recurrence relation

$$K_{\ell+1}(z) = K_{\ell-1}(z) + \frac{2\ell}{z} K_\ell(z). \quad (\text{C}\cdot 9)$$

First, the setting of  $\mathbf{a}_p^\mu = \theta_p^\mu$  leads us to the metric matrix  $\eta_0^{\alpha\beta}$  given by

$$\eta_0^{\mu\nu} = n T^2 \left[ (a_3 \cos \theta + z a_2 \sin \theta) u^\mu u^\nu + \left( (z^2 a_1 - a_3) \cos \theta + z (z^2 a_0 - a_2) \sin \theta \right) \frac{1}{3} \Delta^{\mu\nu} \right], \quad (\text{C}\cdot 10)$$

$$\eta_0^{\mu 4} = \eta_0^{4\mu} = n T^2 z (a_2 \cos \theta + z a_1 \sin \theta) u^\mu, \quad (\text{C}\cdot 11)$$

$$\eta_0^{44} = n T^2 z^2 (a_1 \cos \theta + z a_0 \sin \theta). \quad (\text{C}\cdot 12)$$

By a tedious but straightforward manipulation, we have

$$\eta_{0\mu\nu}^{-1} = (n T^2)^{-1} \left( A(\theta) u_\mu u_\nu + B(\theta) \Delta_{\mu\nu} \right), \quad (\text{C}\cdot 13)$$

$$\eta_{0\mu 4}^{-1} = \eta_{04\mu}^{-1} = (n T^2)^{-1} C(\theta) u_\mu, \quad (\text{C}\cdot 14)$$

$$\eta_{044}^{-1} = (n T^2)^{-1} D(\theta), \quad (\text{C}\cdot 15)$$

where

$$A(\theta) \equiv \frac{a_1 \cos \theta + a_0 z \sin \theta}{(a_3 a_1 - a_2^2) \cos^2 \theta + (a_3 a_0 - a_2 a_1) z \sin \theta \cos \theta + (a_2 a_0 - a_1^2) z^2 \sin^2 \theta}, \quad (\text{C}\cdot 16)$$

$$B(\theta) \equiv \frac{3}{(z^2 a_1 - a_3) \cos \theta + (z^2 a_0 - a_2) z \sin \theta}, \quad (\text{C}\cdot 17)$$

$$C(\theta) \equiv -\frac{1}{z} \frac{a_2 \cos \theta + a_1 z \sin \theta}{(a_3 a_1 - a_2^2) \cos^2 \theta + (a_3 a_0 - a_2 a_1) z \sin \theta \cos \theta + (a_2 a_0 - a_1^2) z^2 \sin^2 \theta},$$

$$(C.18)$$

$$D(\theta) \equiv \frac{1}{z^2} \frac{a_3 \cos \theta + a_2 z \sin \theta}{(a_3 a_1 - a_2^2) \cos^2 \theta + (a_3 a_0 - a_2 a_1) z \sin \theta \cos \theta + (a_2 a_0 - a_1^2) z^2 \sin^2 \theta}. \quad (C.19)$$

Next, the inner product  $\langle \varphi_0^\alpha, \tilde{\varphi}_1^{\mu\beta} \rangle$  is evaluated to be

$$\langle \varphi_0^a, \tilde{\varphi}_1^{\mu b} \rangle = n T^2 \left( a_3 u^a u^\mu u^b + (z^2 a_1 - a_3) \frac{1}{3} (u^a \Delta^{\mu b} + u^\mu \Delta^{ba} + u^b \Delta^{a\mu}) \right), \quad (C.20)$$

$$\langle \varphi_0^a, \tilde{\varphi}_1^{\mu 4} \rangle = n T^2 \left( z a_2 u^a u^\mu + z (z^2 a_0 - a_2) \frac{1}{3} \Delta^{a\mu} \right), \quad (C.21)$$

$$\langle \varphi_0^4, \tilde{\varphi}_1^{\mu b} \rangle = n T^2 \left( z a_2 u^\mu u^b + z (z^2 a_0 - a_2) \frac{1}{3} \Delta^{\mu b} \right), \quad (C.22)$$

$$\langle \varphi_0^4, \tilde{\varphi}_1^{\mu 4} \rangle = n T^2 z^2 a_1 u^\mu. \quad (C.23)$$

Finally, combining  $\eta_{0\alpha\beta}^{-1}$  in Eq.'s (C.13)-(C.15) and  $\langle \varphi_0^\alpha, \tilde{\varphi}_1^{\mu\beta} \rangle$  in Eq.'s (C.20)-(C.23) together with

$$\begin{aligned} \varphi_{1p}^{\mu\alpha} &= [Q_0 \tilde{\varphi}_1^{\mu\alpha}]_p = \tilde{\varphi}_{1p}^{\mu\alpha} - [P_0 \tilde{\varphi}_1^{\mu\alpha}]_p \\ &= \frac{1}{p \cdot \theta_p} \left( p^\mu \varphi_{0p}^\alpha - (p \cdot \theta_p) \varphi_{0p}^\beta \eta_{0\beta\gamma}^{-1} \langle \varphi_0^\gamma, \tilde{\varphi}_1^{\mu\alpha} \rangle \right), \end{aligned} \quad (C.24)$$

we have

$$\varphi_{1p}^{\mu\alpha} = \begin{cases} \frac{1}{p \cdot \theta_p} \left[ \Pi_p \left( u^\mu u^\nu Y_1(\theta) - \Delta^{\mu\nu} Y_2(\theta) \right) \right. \\ \quad \left. + (J_p^\mu u^\nu + J_p^\nu u^\mu) Y_3(\theta) + \pi_p^{\mu\nu} \right] & \text{for } \alpha = \mu, \\ \frac{1}{p \cdot \theta_p} \left[ \Pi_p u^\mu Z_1(\theta) + J_p^\mu Z_2(\theta) \right] & \text{for } \alpha = 4. \end{cases} \quad (C.25)$$

Here, we have introduced the following quantities

$$\Pi_p \equiv \frac{(a_2 a_0 - a_1^2) z^2 (p \cdot u)^2 - (a_3 a_0 - a_2 a_1) z m (p \cdot u) + (a_3 a_1 - a_2^2) m^2}{-3 (a_3 a_1 - a_2^2)}, \quad (C.26)$$

$$J_p^\mu \equiv \frac{\Delta^{\mu\nu} p_\nu \left[ (z^2 a_0 - a_2) z (p \cdot u) - (z^2 a_1 - a_3) m \right]}{-(z^2 a_0 - a_2) z}, \quad (C.27)$$

$$\pi_p^{\mu\nu} \equiv \Delta^{\mu\nu\rho\sigma} p_\rho p_\sigma, \quad (C.28)$$

$$Y_1(\theta) \equiv \frac{-3 (a_3 a_1 - a_2^2) \sin^2 \theta}{(a_3 a_1 - a_2^2) \cos^2 \theta + (a_3 a_0 - a_2 a_1) z \sin \theta \cos \theta + (a_2 a_0 - a_1^2) z^2 \sin^2 \theta}, \quad (C.29)$$

$$Y_2(\theta) \equiv \frac{-(a_3 a_1 - a_2^2) (\cos^2 \theta - \sin^2 \theta)}{(a_3 a_1 - a_2^2) \cos^2 \theta + (a_3 a_0 - a_2 a_1) z \sin \theta \cos \theta + (a_2 a_0 - a_1^2) z^2 \sin^2 \theta}, \quad (\text{C}\cdot 30)$$

$$Y_3(\theta) \equiv \frac{-(z^2 a_0 - a_2) z \sin \theta}{(z^2 a_1 - a_3) \cos \theta + (z^2 a_0 - a_2) z \sin \theta}, \quad (\text{C}\cdot 31)$$

$$Z_1(\theta) \equiv \frac{3 (a_3 a_1 - a_2^2) \cos \theta \sin \theta}{(a_3 a_1 - a_2^2) \cos^2 \theta + (a_3 a_0 - a_2 a_1) z \sin \theta \cos \theta + (a_2 a_0 - a_1^2) z^2 \sin^2 \theta}, \quad (\text{C}\cdot 32)$$

$$Z_2(\theta) \equiv \frac{(z^2 a_0 - a_2) z \cos \theta}{(z^2 a_1 - a_3) \cos \theta + (z^2 a_0 - a_2) z \sin \theta}. \quad (\text{C}\cdot 33)$$

Substituting  $a_0$ ,  $a_1$ ,  $a_2$ , and  $a_3$  in Eq.'s (C·2)-(C·7) into the above equations, we arrive at the explicit representation of  $\varphi_{1p}^{\mu\alpha}$  shown in Eq.(4·11).

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